Outlining Cross Spectral Analysis

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Overview

- Recalling spectral analysis
- Cross Spectral Analysis - Measures
- Cross Spectral Analysis - Interpretation
- Cross Spectral Analysis - Estimation

Spectral Analysis - Measures

Spectral density function

Given a univariate process

$$X_t$$

with covariance function

$$\gamma(h) = E(X_{t+h}X_t)$$

Then the spectral density function is defined as follows:

$$\mathbf{S}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\omega}$$

Periodogram

Given a univariate time series X_t with $t = 1..N$ and with sample covariance function

$$\hat{\gamma}(h) = \frac{1}{N} \sum X_{t+h} X_t,$$

a first guess estimator for the spectrum is the periodogram:

$$p(\omega_j) = \sum_{|k|<N} \hat{\gamma}(k) e^{-ik\omega_j}$$

or equivalently (for an easier computation)

$$p(\omega_j) = F(\omega_j) F^*(\omega_j),$$

where $F(\omega_j)$ denotes the discrete Fourier trafo of X_t and $\omega_j = \frac{2\pi j}{N}$.

Spectral Analysis - Estimation

Frequency resolution

$$\Delta\omega = \frac{2\pi}{N}$$

Nyquist frequency

$$f_{NY} = \frac{1}{2}, \text{for general } \Delta t : \quad f_{NY} = \frac{1}{2\Delta t}$$

Properties of the periodogram

The single values of the periodogram are χ^2-distributed with mean and variance proportional to the real spectrum $S(\omega)$, but independent of N. Thus, the periodogram is an unbiased but not consistent estimator of the true spectrum. When N tends to infinity, the frequency-resolution $\Delta\omega$ will tend to infinity, but the variance at every single frequency will remain constant.

Cross Spectral Analysis - Measures

Given a bivariate process

$$\mathbf{X}_t = (X_{t1}, X_{t2})^T$$

with covariance function matrix

$$\Gamma(h) = (\gamma_{ij}(h)) = (E(X_{t+h,i}X_{tj})) \quad , \text{with} \quad i,j = 1,2$$

Then the spectral density matrix is defined as follows:

$$\mathbf{S}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \Gamma(h) e^{-ih\omega} = \begin{pmatrix} s_{11}(h) & s_{12}(h) \\ s_{21}(h) & s_{22}(h) \end{pmatrix}$$

$$\text{with} \quad s_{21}(\omega) = {s_{12}}^*(\omega)$$

Cross Spectral Analysis - Measures

$$s_{12}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_{12}(h) e^{-ih\omega}$$

is called the cross spectrum, which can be separated into amplitude spectrum and phase spectrum:

$$s_{12}(\omega) = A(\omega) e^{i\Phi(\omega)} \quad \text{with} \quad \Phi(\omega) \in (-\pi, \pi]$$

The coherence is defined as

$$K(\omega) = \frac{|s_{12}(\omega)|}{(s_{11}(\omega) s_{22}(\omega))^{1/2}} \quad , \text{with} \quad 0 \leq K \leq 1$$

and measures the linear relation at frequency ω.

Linear time invariant filter I

If one process is a linearily filtered version of the other

$$\mathbf{X}_{t2} = \sum_{h=-\infty}^{\infty} \Psi_h \mathbf{X}_{t-h,1}$$

then it follows immediately that

$$K(\omega) \equiv 1$$

Cross Spectral Analysis - Interpretation

Linear time invariant filter II

If both processes are linearily filtered versions of two other processes

$$\mathbf{Y}_{t1} = \sum_{h=-\infty}^{\infty} \alpha_h \mathbf{X}_{t-h,1}$$

$$\mathbf{Y}_{t2} = \sum_{h=-\infty}^{\infty} \beta_h \mathbf{X}_{t-h,2}$$

then the coherence between the filtered processes equals that of the unfiltered:

$$K_Y(\omega) \equiv K_X(\omega)$$

Cross Spectral Analysis - Interpretation

Time delay

If one process is a delayed version of the other

$$\mathbf{X}_{t2} = \mathbf{X}_{t-d,1} + N_t$$

then the delay is given by the slope of the phase spectrum

$$\partial_\omega \Phi(\omega) = d$$

In general, the phase spectrum measures the phase lag of $\mathbf{X}_{t2}$ behind $\mathbf{X}_{t2}$ at frequency ω.

The Cross Periodogram

If the discrete Fourier Transformation of a bivariate time series $\mathbf{X}_t = (X_{t1}, X_{t2})^T$ is given as

$$\mathbf{F}(\omega_j) = N^{-1/2} \sum_{t=1}^{N} \mathbf{X}_t e^{-it\omega_j}$$

then the periodogram is a first guess estimator for the spectrum:

$$\mathbf{P}(\omega_j) = \sum_{|k|<N} \hat{\mathbf{\Gamma}}(k) e^{-ik\omega_j}$$

or equivalently (for an easier computation)

$$\mathbf{P}(\omega_j) = \mathbf{F}(\omega_j)\mathbf{F}^*(\omega_j)$$

This estimator is unbiased, but again not consistent.

Smoothed periodogram

Using a smoothing kernel with width m in the frequency domain,

$$\hat{S}(\omega_j) = \frac{1}{2\pi} \sum_{|k|<m_N} W_N(k)\mathrm{P}(\omega_{j+k})$$

one is able to construct an asymptotically unbiased estimator

$$\lim_{N\to\infty} E\hat{f}(\omega) = f(\omega), \quad \text{when} \quad m/N \to 0 \quad \text{for} \quad N \to \infty$$

which is also consistent.

Real and imaginary part of the cross spectrum are called Co- and Quadraturspectrum:

$$\hat{c}(\omega_j) = \tfrac{1}{2}(\hat{s}_{12}(\omega_j) + \hat{s}_{21}(\omega_j))$$
$$\hat{q}(\omega_j) = \tfrac{1}{2}(\hat{s}_{12}(\omega_j) - \hat{s}_{21}(\omega_j))$$

Cross Spectral Analysis - Estimation

Cross amplitude spectrum

The estimator for the Cross amplitude spectrum

$$\hat{A}(\omega_j) = \left(\hat{c}^2(\omega_j) + \hat{q}^2(\omega_j)\right)^{1/2}$$

is asymptotically normally distributed with vanishing variance

$$\hat{A}(\omega_j) \quad \text{is} \quad AN\left(A(\omega_j), \left[\sum_{|k|<m_N} W_N^2(k)\right] A^2(\omega_j)(\frac{1}{K^2} + 1)/2\right)$$

NB! For low coherency K, the variance gets large!

Phase spectrum

The estimator for the phase spectrum

$$\hat{\Phi}(\omega_j) = \arg(\hat{c}(\omega_j) - i\hat{q}(\omega_j))$$

is also asymptotically normally distributed with vanishing variance

$$\hat{\Phi}(\omega_j) \quad \text{is} \quad AN\left(\Phi(\omega_j), \left[\sum_{|k|<m_N} W_N^2(k)\right] A^2(\omega_j)(\frac{1}{K^2} - 1)/2\right)$$

Also here, for low coherency K, the variance gets large!

Coherency

The coherency ist estimated as

$$\hat{K}(\omega_j) = \sqrt{\frac{\hat{c}^2(\omega_j) + \hat{q}^2(\omega_j)}{\hat{s}^2_{11}(\omega_j)\hat{s}^2_{11}(\omega_j)}}$$

It is also asymptotically normally distributed with vanishing variance

$$\hat{K}(\omega_j) \quad \text{is} \quad AN\left(K(\omega_j), \left[\sum_{|k|<m_N} W_N^2(k)\right](1 - K^2(\omega_j))/2\right)$$

Also here, the variance gets large for low real coherency.

Cross Spectral Analysis - Estimation

Test against zero coherency

Often it is of interest, if the estimated coherency is compatible with zero. If $W_n(k) = (2m+1)^{-1}$ for $|k| \leq m$ and $W_n(k) = 0$ otherwise, then the measure

$$Y = \frac{2m\hat{K}^2}{1-\hat{K}^2}$$

is F-distributed under the Null hypothesis $K(\omega) = 0$:

$$Y > F_{1-\alpha}(2, 4m)$$

A test against zero cross spectrum?

Often, one finds in literature tests for “signficant cross spectra”, especially in wavelet spectral analysis. However, when one starts to think about distributions under the Null hypothesis, one gets into logical difficulties:
As we are only dealing with estimators derived from finite data, the estimated cross amplitude spectrum $\hat{A}$ will always be larger than zero. A test would have to quantify, if this deviation from zero would be significant. However, as the cross amplitude spectrum is not normalized, the difference could be large because either the one spectrum is large or the other (or both), even if there is no correlation. Thus, in principle, such a test is invalid and produces false positive results.

Literature

P.J. Brockwell and R.A. Davis, Time Series: Theory and Methods, Springer Series in Statistics, Springer, 1987.