Delay Estimation and the Estimation of Coherence and Phase

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Abstract—The method of *delay* estimation given in Hamon and Hannan [1] is reviewed and a modification involving *autoregressive model* fitting is proposed. It is shown how numerical problems associated with the optimization of the criterion used in both the original and modified procedures can be circumvented. The modified procedure and a procedure due to Chan, Riley, and Plant [2] are compared to the Hamon-Hannan procedure by *simulations*. For moderate to large signal-to-noise ratios, the modified Hamon-Hannan procedure appears to provide better estimates. In the case of low signal-to-noise ratios, with the smaller size, the Chan, Riley, and Plant procedure performed best.

I. Introduction

CONSIDER two recorders receiving a delayed form of a common signal $s(t)$ together with noise. The signal is assumed to be a zero mean *stationary process* and thus has a spectral representation [5, p. 41]

$$s(t) = \int_{-\infty}^{\infty} e^{-jt\omega}\, dz(\omega) \qquad (-\infty < t < \infty).$$

We assume that $s(t)$ has an absolutely continuous spectrum with spectral density $G_s(\omega)$. The processes received at the two recorders are

$$x(t) = s(t) + n_1(t),\ y(t) = \int_{-\infty}^{\infty} e^{-j(t - D(\tau,\omega))\omega} a(\omega)\, dz(\omega) + n_2(t) \tag{1}$$

where $D(\tau, \omega)$ is the relative *delay*, $a(\omega)$ is the attenuation relative to the first recorder, and the $n_j(t)$ $(j = 1, 2)$ are zero mean *stationary* noise *processes* incoherent with each other and $s(t)$. $D(\tau, \omega)$ is assumed to be a known function of parameters τ_1, $\cdots$, τ_r and ω. In the *nondispersive case* $D(\tau, \omega) = \tau$ where τ is the pure *delay*.

There is a considerable body of literature concerning the estimation of *delay* (see [4] for some references). Here we review the procedure in [1] and introduce a procedure based on *autoregressive model* fitting. These are together compared with a procedure in [2]. A numerical problem associated with the procedure in [1], and the modified procedure, is discussed.

We assume that $x(t)$, $y(t)$ have been sampled at equal time intervals, that this interval is the time unit which is sufficiently small for aliasing effects to be ignored. If the $n_i(t)$ $(i = 1, 2)$ have spectral densities $G_i(\omega)$ $(i = 1, 2)$, the spectral densities of $x(t), y(t)$ are $G_x(\omega) = G_s(\omega) + G_1(\omega), G_y(\omega) = |a(\omega)|^2 G_s(\omega) + G_2(\omega)$ and the cross-spectral density is $G_{yx}(\omega) = a(\omega)G_s(\omega)$ $\exp jD(\tau, \omega)\omega$, $0 \leqslant \omega \leqslant \pi$. Then the coherence and phase at frequency ω are $\sigma(\omega) = |G_{yx}(\omega)|/\{G_x(\omega)G_y(\omega)\}^{1/2}$, $\phi(\omega) = \omega D(\tau, \omega)$.

Manuscript received March 4, 1980; revised September 26, 1980.

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II. The Procedures

A detailed account of the Hamon-Hannan procedure (hereafter referred to as HH) is given in [1]. Estimates $\hat{\sigma}$, $\hat{\phi}$ are formed at the M frequencies $\lambda_u = (2u - 1)\pi/(2M)$ $(u = 1, \cdots, M)$. Then the vector of delay parameters is estimated by $\hat{\tau}$, which maximizes

$$Q_H(\tau) = \frac{1}{M} \sum_{\lambda_u \in B} \hat{W}(\lambda_u) \cos\left(\hat{\phi}(\lambda_u) - D(\tau, \lambda_u)\lambda_u\right). \tag{2}$$

Here $\hat{W}(\lambda_u) = \hat{\sigma}^2(\lambda_u)/(1 - \hat{\sigma}^2(\lambda_u))$ estimates $W(\lambda_u) = \sigma^2(\lambda_u)/(1 - \sigma^2(\lambda_u))$. M and the band B are chosen by the experimenter. The estimates of $\hat{\sigma}$, $\hat{\phi}$ can be formed from the discrete Fourier transforms

$$w_x(\omega_k) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} x(t) \exp jt\omega_k,\ w_y(\omega_k) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} \cdot\ y(t) \exp jt\omega_k$$

where $\omega_k = 2\pi k/T$ $(k = 1, \cdots, [\frac{1}{2}T])$ and T is the number of data points. A fader could be used to compute $w_x(\omega_k), w_y(\omega_k)$. (See [1], [5] for details.) Unver general conditions $\hat{\tau}$ converges (almost surely) to τ and $\sqrt{T(\hat{\tau} - \tau)}$ has an asymptotic limiting normal distribution with zero means and covariance matrix V^{-1} where V is estimated by

$$\hat{V}_{jk} = M^{-1} \sum_{\lambda_u \in B} \lambda_u^2 \hat{W}(\lambda_u) \frac{\partial D(\tau, \lambda_u)}{\partial \tau_j} \frac{\partial D(\tau, \lambda_u)}{\partial \tau_k}.$$

The weight function, $W(\lambda_u)$, is optimal in that it makes the covariance matrix V a minimum (see [1] and [6]). As pointed out in [1], $\hat{\sigma}$ may tend to be much too high when σ is low so that frequencies containing little information are given too much weight. The following procedure is suggested as a means of improving the estimate of W. If

$$\begin{bmatrix} G_x(\omega) & G_{xy}(\omega) \\ G_{yx}(\omega) & G_y(\omega) \end{bmatrix} \simeq \frac{1}{2\pi} \left[\sum_{i=0}^{p} A_i e^{ji\omega}\right]^{-1} \cdot K \left[\sum_{i=0}^{p} A_i e^{-ji\omega}\right]^{-1} \tag{3}$$

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where the A_i are 2×2 matrices and K is the covariance matrix of the "innovations," then estimates of p, K and the A_i will yield estimates of G_x, G_y and G_{xy} which yield estimates of σ and W. Since such estimates (say $\tilde{\sigma}$) are available for any frequency ω, an estimate of the vector of delay parameters is $\hat{\tau}$, which maximizes

$$\tilde{Q}_H(\tau) = \frac{1}{N} \sum_{0 < \omega_k < \pi} \tilde{W}(\omega_k) \cos\left(\hat{\phi}(\omega_k) - D(\tau, \omega_k)\omega_k\right) \tag{4}$$

where $\tilde{W}(\omega_k) = \tilde{\sigma}^2(\omega)/\{(1 - \tilde{\sigma}^2(\omega)\}$. *Now* $\hat{\phi}(\omega_k)$ *is the argument of* $w_y(\omega_k)\overline{w_x(\omega_k)}$. There is a large literature on the fitting of finite parameter models such as *autoregressive* (AR) *models*, i.e., of the form (3), (see [7]) and on the determination of the order p. One technique (see [3]) consists of fitting (3) to the data for a range of orders p and then selecting the p to minimize

$$\mathrm{AIC}(p) = \log \det \hat{K}_p + 8p/T. \tag{5}$$

Here $\hat{K}_p$ is the estimated residual covariance matrix for the fitted model of order p (see also [8] and [9]). The method of [10] involving the recursive solution of the Yule-Walker equations provides a computationally efficient method for determining estimates of the A_j and K in (3).

For $D(\tau, \omega) = \tau$ the method of Chan, Riley, and Plant [2] (here called the CRP procedure) involves the fitting of a distributed lag relationship between $y(t)$ and $x(t)$ of the form

$$y(t) = \sum_{-p}^{p} \zeta_i x(t - i) + \epsilon(t)$$

where $\epsilon(t)$ is a *stationary* noise *process*. Estimates of the ζ_i (say $\hat{\zeta}_i$) are obtained by ordinary least squares and the estimate of the delay is then determined as the value $\hat{\tau}$ maximizing

$$Q_{\mathrm{CRP}}(\tau) = \frac{1}{2p+1} \sum_{-p}^{p} \hat{\zeta}_j \frac{\sin \pi(\tau - j)}{\pi(\tau - j)}. \tag{6}$$

The criterion (6) is not, in general, symmetric with regard to the choice of regressor and regressand. If p is allowed to increase with T it is said in [2] that this criterion is asymptotically equivalent to the criterion given by (4) with $\tilde{W}(\omega_k)$ replaced by an estimate of $\sigma(\omega_k)\sqrt{G_y(\omega_k)/G_x(\omega_k)}$. Thus, this procedure then yields an estimator of τ whose asymptotic variance exceeds that of the $\hat{\tau}$ obtained using either (2) or (4).

If $W(\omega)$ were known, the HH procedure would be close to a maximum likelihood procedure. It is for this reason that we have sought to improve the estimation $W(\omega)$ by the use of a finite parameter model. The CRP procedure uses a finite parameter model also (effectively) to estimate phase, but the weighting of frequencies in this procedure is not optimal. A third alternative not investigated here would be to use a finite parameter model also for the noise in the distributed lag relationship so as to recover some of this loss due to incorrect weighting of frequencies.

III. Simulations: Pure Delay

To compare the three procedures in the pure *delay* case a *simulation* study was carried out with $T = 1024$ and 256. The

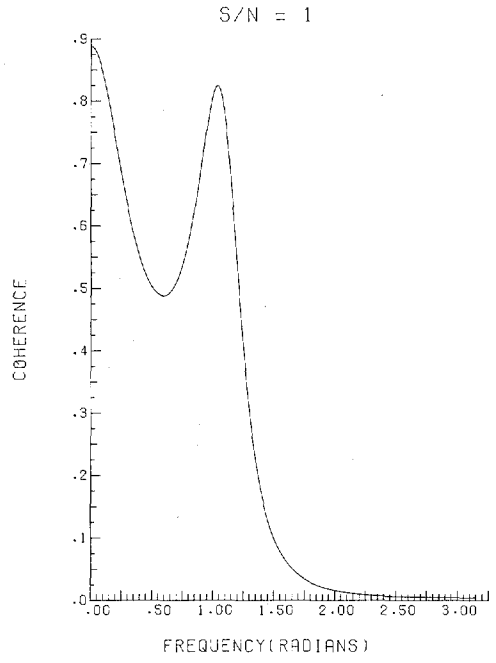

Fig. 1. Coherence. Relatively high signal-to-noise ratio.

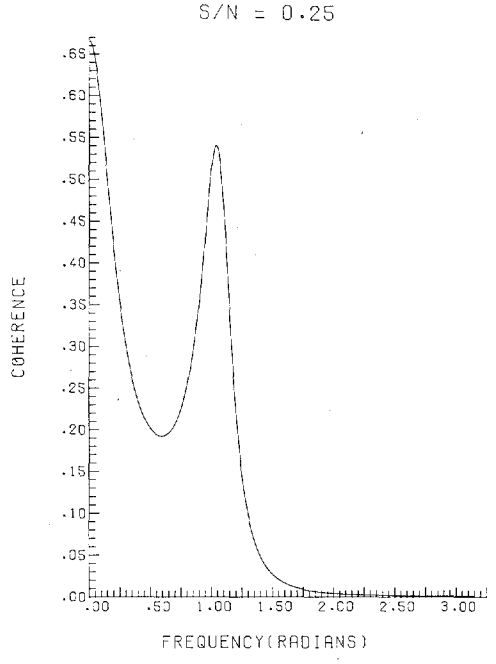

Fig. 2. Coherence. Relatively low signal-to-noise ratio.

data were generated according to (1) with $a(\omega) = 1$, $G_1(\omega) = G_2(\omega) = \alpha^2/(2\pi)$ and $s(t)$ was generated via an autoregression with spectral density

$$G_s(\omega) = \frac{1}{2\pi} |1 - 1.77 \exp j\omega + 1.593 \exp j2\omega - 0.7047 \cdot \exp j3\omega|^{-2}.$$

The two values of τ were $\tau = 0.5$, $\tau = 10.5$. The S/N ratios (the ratio of the signal variance to the noise variance) were 1 and 0.25, i.e., $\alpha^2 = 8.97995, 35.9198$, respectively. The coherence function is shown in Figs. 1 and 2.

For the HH procedure m, the number of ω_k in the band at λ_u, was 9 for $T = 1024$ and 5 for $T = 256$. Thus, M was effec-

TABLE I
NONDISPERSIVE CASE; SAMPLE SIZE = 1024

S/N	τ	Hamon-Hannan			Modified Hamon-Hannan			Chan-Riley-Plant		
		Sample mean	Sample std. dev.	Theoretical std. dev.	Sample mean	Sample std. dev.	Theoretical std. dev.	Sample mean	Sample std. dev.	Theoretical std. dev.
1	0.5	0.4609	0.1404	0.0708	0.4971	0.1220	0.0708	0.4576	0.2651	0.2846
1	10.5	10.4719	0.1472	0.0708	10.4773	0.1221	0.0708	10.5023	0.2391	0.2846
0.25	0.5	0.4131	0.3802	0.1735	0.5541	0.2861	0.1735	0.3362	0.3802	0.6581
0.25	10.5	10.4488	0.3608	0.1735	10.4696	0.3309	0.1735	10.4616	0.3696	0.6581

TABLE II
NONDISPERSIVE CASE; SAMPLE SIZE = 256

S/N	τ	Hamon-Hannan			Modified Hamon-Hannan			Chan-Riley-Plant		
		Sample mean	Sample std. dev.	Theoretical std. dev.	Sample mean	Sample std. dev.	Theoretical std. dev.	Sample mean	Sample std. dev.	Theoretical std. dev.
1	0.5	0.5588	0.3363	0.1416	0.5032	0.2146	0.1416	0.5875	0.4719	0.5693
1	10.5	10.4312	0.3812	0.1416	10.4945	0.3460	0.1416	10.3961	0.3894	0.5693
0.25	0.5	0.6749	0.5893	0.3470	0.6904	0.5916	0.3470	0.6287	0.6599	1.3163
0.25	10.5	10.6070	0.6989	0.3470	10.3732	0.5800	0.3470	10.4788	0.4884	1.3163

tively 56 and 25. A taper was not used and may have helped the HH procedure. For the modified HH procedure, p was selected using (5) for $5 \leqslant p \leqslant 20$ for $\tau = 0.5$; $10 \leqslant p \leqslant 20$ for $\tau = 10.5$. In general, p should be large enough to ensure adequate resolution of the two component spectra and the cross-spectrum of the observed process. For large τ (e.g., $\tau = 10.5$), it is not hard to see that an order of at least the delay τ should be chosen. For the CRP procedure, similar considerations apply. For the CRP procedure, p was 5 ($\tau = 0.5$) and 15 ($\tau = 10.5$). The results of the simulations given were based on 20 independent replications.

For $T = 1024$, the results are given in Table I. The modified HH procedure performed best, followed by the HH procedure. However, the HH procedures gave sample standard deviations (sd's) that were significantly greater than the theoretical (asymptotic) values. Indeed, the smallest ratio of the former to the latter was 1.6, which is significant at any reasonable level. For the CRP procedure, the sample sd's were not significantly different ($S/N = 1$) or significantly less than ($S/N = 0.25$) the theoretical values.

For $T = 256$ the results are as given in Table II. Now, for $S/N = 1$ in terms of sample sd's (and mean-square errors), the modified HH procedure performed best, followed by the HH procedure, but for $S/N = 0.25$ the CRP procedure performed best, followed by the modified HH procedure.

The following may explain these results. The band B was taken as $(0, \pi)$. The theoretical sd for HH does not reflect inaccuracy in $\hat{W}$, since for large enough T this effect will be of lower order. However, the effect may not be negligible, particularly for low σ. Figs. 3 and 4 show $\hat{\phi}$ for two *simulations* and indicate how inaccurate is $\hat{\phi}$ for low σ. (The two $\hat{\tau}$ values were 0.475 for Fig. 3 and 0.543 for Fig. 4.) *Undoubtedly, better results would have been achieved had frequencies above about 1.25 rads been eliminated* (see [1, p. 141]). The theoretical weight function for the HH procedures for $S/N = 1$ is given in Fig. 5. That for the CRP procedure is $\sigma(\omega)$, exhibited for $S/N =$ 1 in Fig. 1. It may be noted that when $\sigma = 0$, $(m - 1)\,\hat{W}$ is (approximately) distributed as F with 2 and $2(m - 1)$ degrees of freedom (see [5, p. 261] for details). For $m = 9$ this must be above 0.22 (i.e., $\hat{\sigma} > 0.16$) for significance even at the 20 percent level.

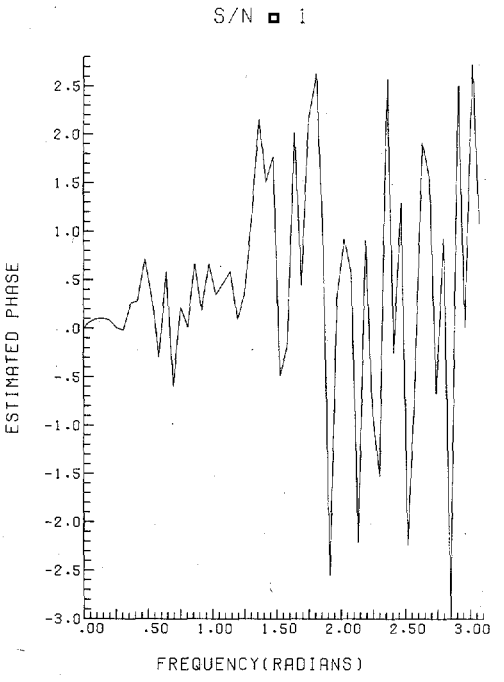

Fig. 3. Estimated phase for a simulation with relatively high signal-to-noise ratio.

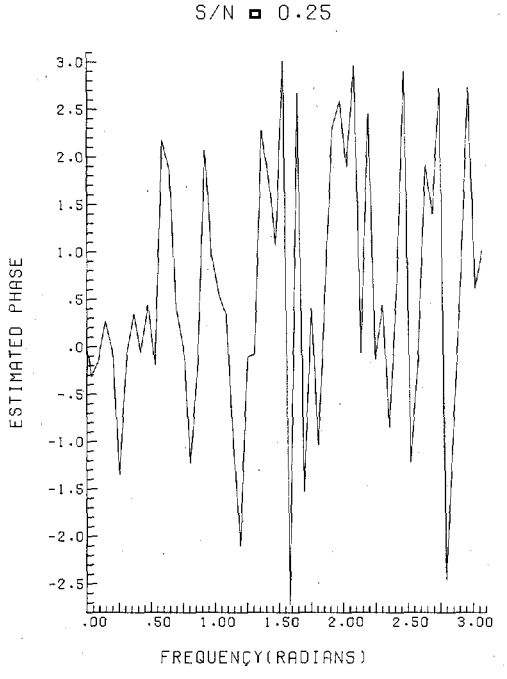

Fig. 4. Estimated phase for a simulation with relatively low signal-to-noise ratio.

The criterion being optimized will approach a limiting form as $T \to \infty$. These forms are shown in Figs. 6 and 7, for $S/N = 1$. Each is shown as a function of $\tau - \tau_0$ where τ_0 is the true value. In each case there is a main lobe and smaller sidelobes. The theoretical variances take no account of the possibility that a sample maximum may be near a sidelobe since, in the limit,

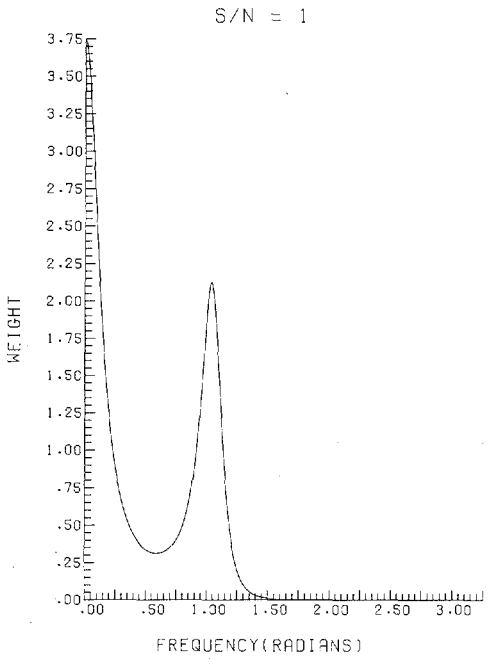

Fig. 5. Weight function $W(\omega)$ for relatively high signal-to-noise ratio.

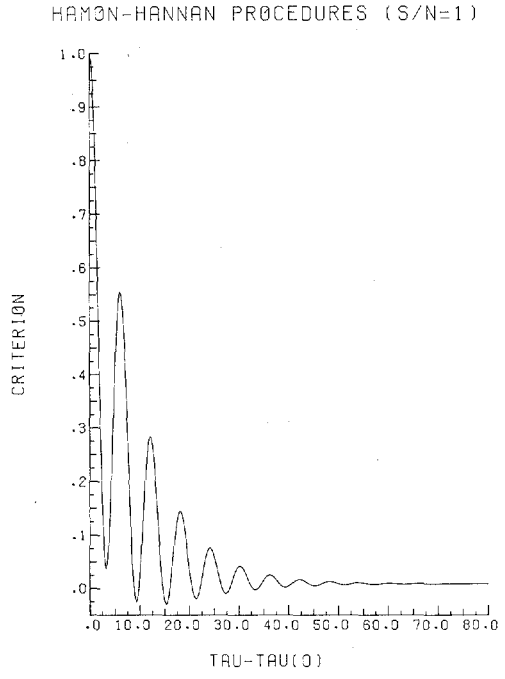

Fig. 6. Limiting form of the HH criterion for relatively high signal-to-noise ratio.

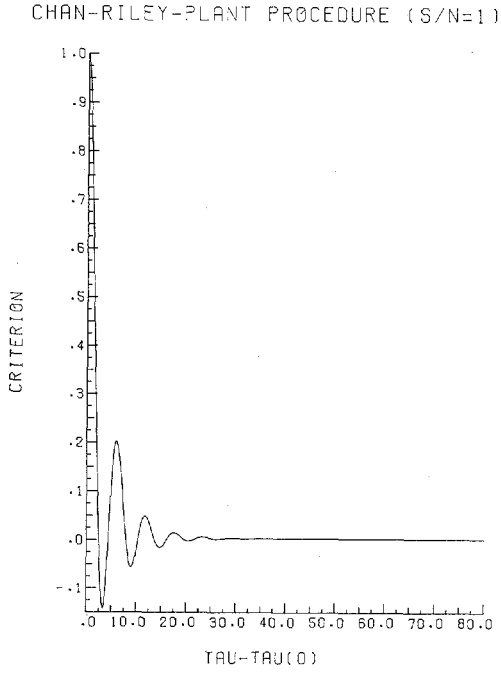

Fig. 7. Limiting form of the CRP criterion for relatively high signal-to-noise ratio.

that will not happen. However, again, it may happen in practice. (See [1, p. 141] for a discussion.) For $T = 1024$, this happened only once for the modified HH procedure, and then it happened for all three procedures. It happened twice for HH. It did not happen at $S/N = 1$ for $T = 256$, but for $T = 256$ and $S/N = 0.25$ it happened a number of times, especially for the modified HH procedure. This was due to the difficulty of fitting G_x, G_y, G_{xy} by a model of the form of (3) for $T = 256$ and α^2 relatively large. (It should be kept in mind that a distributed lag model may not work well except for large p, i.e., large T, when $G_x \neq G_y$ and the noise spectra are not white and equal.) For the HH procedures the false global maxima, when they occurred, were for large $\tau - \tau_0$. The technique of optimization used commenced iterations from τ_0 and thus located the nearest relative maximum to τ_0. Thus there was no inflation of the observed sd's from a false location of a global maximum for the HH procedures. It could be argued that this optimization procedure was reasonable, since values of $\hat{\tau}$ far away from τ_0 would be eliminated as spurious on prior grounds. For CRP there was a tendency for both a global and a relative maximum to be very near to τ_0, and the procedure on three or four occasions chose the relative maximum when $S/N = 0.25$, $T = 256$. This may have contributed to the reduction of the sd for CRP.

There is a problem with the location of the absolute maximum for HH because of the large number of relative maxima. However, this is easily overcome. Consider $Q_H(\tau)$ for example. This has period M, so consider τ of the form $t/2, t = 0, \pm 1, \cdots, \pm M$. Then, for such τ values, $Q_H(\tau)$ is the real part of

$$\frac{1}{T'} \sum_{k=1}^{T'} C_k \exp(j2\pi kt/T'), \qquad T' = 2T \tag{7}$$

where $C_k = T'M^{-1} \hat{W}(\lambda_u) \exp(-j\hat{\theta}(\lambda_u))$ for $\lambda_u = 2\pi k/T$ and zero otherwise. Now (7) may be evaluated *by fast Fourier transformation*, a global maximum located and a standard function optimization routine used to determine $\hat{\tau}$.

IV. Simulations: Dispersive Case

The simulations for the *dispersive case* were for the same S/N ratios, etc., the delay $D(\tau; \omega)$ being $\tau\omega$, so that $\phi(\omega) = \tau\omega^2$, with $\tau = 1.5$ and $T = 1024$. Again, 20 independent simulations were generated. The results are given in Table III. Since the CRP procedure does not apply to the *dispersive case*, the HH procedures only were simulated. The modified HH procedure performed better, but the significant discrepancies between the sample and theoretical sd persisted.

The limiting form of the criterion for $S/N = 1$ is shown in

TABLE III
DISPERSIVE CASE; SAMPLE SIZE = 1024

S/N	τ	Hamon-Hannan Sample mean	Hamon-Hannan Sample std. dev.	Hamon-Hannan Theoretical std. dev.	Modified Hamon-Hannan Sample mean	Modified Hamon-Hannan Sample std. dev.	Modified Hamon-Hannan Theoretical std. dev.
1	1.5	1.5615	0.1633	0.0697	1.5098	0.0914	0.0697
0.25	1.5	1.3542	0.3937	0.1718	1.4124	0.3304	0.1718

Fig. 8. A procedure similar to that based on (7) is again possible. Note that both criteria can be approximated as

$$\frac{1}{\pi}\int_0^{\pi} \bar{W}(\omega)\cos\left(\bar{\phi}(\omega) - \tau\omega^2\right)d\omega$$

where, in the case of the HH procedure, for example, $\bar{W}(\omega)$, $\bar{\phi}(\omega)$ are given by $\hat{W}(\lambda_u)$, $\hat{\phi}(\lambda_u)$ for ω a member of the narrow band containing λ_u. Consider evaluating the criterion over values of $\tau = t/4$ $(t = 0, \pm 1, \cdots, 2T)$. Then the above can be written as

$$\frac{1}{\pi}\int_0^{\pi} \omega^{-1/2}\ \bar{W}(2\sqrt{\omega})\cos t\omega d\omega$$

where we have defined $\bar{W}(\omega)$ to be zero for $\omega > \pi$. But this is approximately the real part of

$$\left\{\frac{1}{T}\sum_{k=1}^{T}(\pi k/T)^{-1/2}\ \bar{W}(2\sqrt{\pi k/T})\exp j2\pi kt/(2T)\right\}$$

which can be evaluated by the fast Fourier transform. This technique can be generalized to deal with any dispersion situation where $D(\tau;\omega) = \tau D(\omega)$ and $D(\omega)$ is a monotone function of ω.

V. Conclusions

These simulations favor the HH procedures, and the modified HH procedure especially, except in the low S/N case for $T = 256$ and $\tau = 10.5$. A number of factors might make these results less relevant to a real estimation situation.

1) In a narrow-band situation, the variation of the coherence across the band might be small and optimal weighting less important.

2) The high experimental, as compared to theoretical, sd for the HH procedures is probably mainly due to overweighting of high frequencies where σ is low. As said in [1, Section 6.2], this could be reduced in effect by aggregating bands where $\hat{\sigma}$ is low with a smaller number of wider bands. It might be better to omit such bands from the calculations. In practice, this problem may be less important, since the frequencies where the signal is sensibly present may be known beforehand.

3) It is possible that, for the CRP procedure, the sample sd is too low because, by starting iterations at the true value, a subsidiary local maximum of the criterion has been formed. In practice, this subsidiary maximum could be at a wrong value, and this emphasizes the usefulness of scanning the criteria. This effect would increase as S/N falls and p rises. However, this may not be the only or major reason for the lower sd's, and an

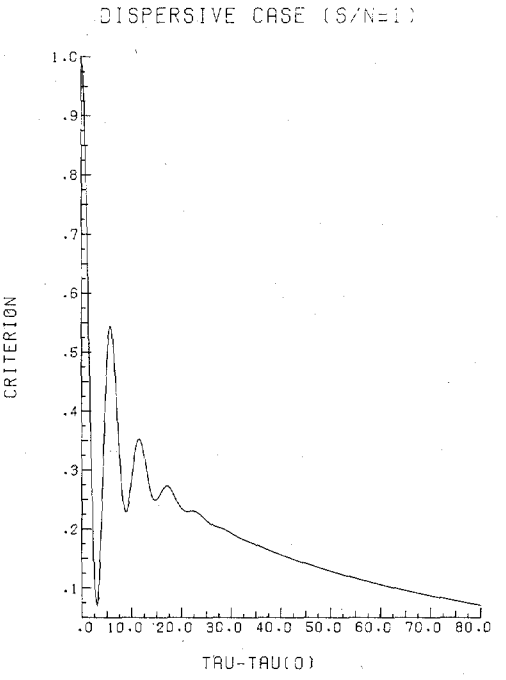

Fig. 8. Limiting form of the HH criterion in the dispersive case for relatively high signal-to-noise ratio.

alternative explanation may be the advantage gained from a simple finite parameter model in the circumstances of the simulations.

4) The HH procedures could easily be misused to locate the estimate near a maximum of a sidelobe. The scanning procedures based on fast Fourier transformation avoid this.

In retrospect, many of these phenomena could have been investigated by a more extensive set of simulations involving a) a weighting of the frequencies with the true (asymptotic) weight function for the HH procedures and b) omission of frequencies above $\pi/2$ for the HH procedures. It also would have been interesting to examine the estimate of $\phi(\omega)$ got from the autoregression procedure and from $\Sigma\hat{\xi}_i \exp ij\omega$, and to examine more realistic models for the noise processes.

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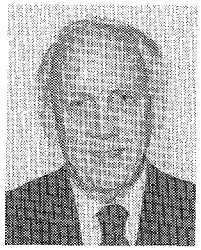
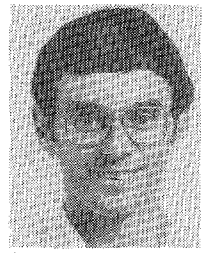


Edward J. Hannan was born in 1921 in Melbourne, Australia. He received the bachelor's degree in commerce (business administration) in 1949.

He received his early education in Melbourne, left school at 15, and worked as a bank clerk until 1940, when he commenced a five-year period of army service in an infantry battalion. In early 1946 he commenced his tertiary education under an assistance scheme for ex-servicemen. After some introduction to elementary mathematics and statistics during his degree, he continued to study mathematics while he worked as an economist with (what is now) the Reserve Bank of Australia. In mid-1953 he was sent by that institution to study for one year at the newly formed Australian National University, Canberra, where he came under the influence of P. A. Moran. He has remained at ANU since and has worked almost entirely in the field of time series analysis, in which he has published two books. His interest in problems of a geophysical nature has arisen from consulting with geophysicists and oceanographers, especially B. V. Hamon. His most recent work has predominantly been in the statistical theory of linear systems.



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Error Analysis of Time Delay Estimation Using a Finite Integration Time Correlator

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Abstract—**A methodology is presented for analyzing the error associated with time delay estimation using a finite integration time correlator, processing waveforms received at two separate sensors. The type of signal considered is a sinusoid whose amplitude is randomly modulated. The signals are assumed to be imbedded in additive Gaussian noise. Before they are correlated, the received waveforms are converted to a lower center frequency by mixers whose local oscillators are assumed to contain phase noise. By direct calculation in the time domain, the variance of the error in the time delay estimate is shown to be a function of integration time, signal-to-noise ratios, signal and noise bandwidths, and phase noise variance. The phase noise is shown to limit the accuracy of the time delay estimate. However, without phase noise, using the methodology the accuracy is shown to approach that obtained by the maximum likelihood estimator.**

Manuscript received May 23, 1980; revised October 22, 1980. This work was supported by the Naval Research Laboratory under Contract N00173-79-C-0473.

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I. INTRODUCTION

SUPPOSE that a transmission from a single source is received at two separate sensors. If the difference in the times of arrival of the transmission, or "time difference of arrival" (TDOA), can be measured or estimated, then inferences can be made concerning the location of the source. Location in three dimensions, of course, requires four measurements, and the localization accuracy is a function of the placement of the sensors as well as of the accuracy in TDOA.

In this paper we calculate the performance of a particular system for measuring TDOA—a finite time cross correlator—

0096-3518/81/0600-0490$00.75