The Importance of Phase in Signals

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Invited Paper

Abstract—In the Fourier representation of signals, spectral magnitude and phase tend to play different roles and in some situations many of the important features of a signal are preserved if only the phase is retained. Furthermore, under a variety of conditions, such as when a signal is of finite length, phase information alone is sufficient to completely reconstruct a signal to within a scale factor. In this paper, we review and discuss these observations and results in a number of different contexts and applications. Specifically, the intelligibility of phase-only reconstruction for images, speech, and crystallographic structures are illustrated. Several approaches to justifying the relative importance of phase through statistical arguments are presented, along with a number of informal arguments suggesting reasons for the importance of phase. Specific conditions under which a sequence can be exactly reconstructed from phase are reviewed, both for one-dimensional and multidimensional sequences, and algorithms for both approximate and exact reconstruction of signals from phase information are presented. A number of applications of the observations and results in this paper are suggested.

I. Introduction

In the Fourier representation of signals, spectral magnitude and phase tend to play different roles and in some situations, many of the important features of a signal are preserved if only the phase is retained. A corresponding statement cannot in general be made for the spectral magnitude. This observation about phase has been made in a number of different contexts and applications and including one-dimensional, two-dimensional and three-dimensional signals. For example, both phase-only and magnitude-only acoustical and optical holograms have been studied. For phase-only holograms (also referred to as kinoforms) only the phase of the scattered wavefront is recorded and the magnitude is assumed to be constant while in the magnitude-only hologram the phase is assumed to be zero and only the magnitude of the scattered wavefront is recorded. In general, with reconstruction from magnitude-only holograms, the reconstructed object is not of much value in representing the original object whereas reconstructions from phase-only holograms have many important features in common with the original objects. Closely related to phase-only and magnitude-only holograms are phase-only and magnitude-only images. As with kinoforms, a phase-only image has Fourier transform phase equal to that of the original image and a Fourier transform magnitude of unity or perhaps more generally representative of the spectral magnitude of images such as the average over an ensemble of unrelated images. As is demonstrated by the examples in Section II, many of the features of the original image are clearly identifiable in the phase-only image but not in the magnitude-only image. Similar observations have also been made in the context of speech signals and X-ray crystallography. Specifically, for speech it has been shown that the intelligibility of a sentence is retained if the phase of the Fourier transform of a long segment of speech is combined with unity magnitude. In the context of X-ray crystallography, details of the crystallographic structure are often inferred from X-ray diffraction data. The Fourier synthesis of the structure from only the correct magnitude of the diffraction data with zero phase in general does not preserve the atomic structure whereas Fourier synthesis using only the correct phase with unity magnitude does reflect the correct atomic structure. These examples, elaborated on in Section II, suggest very strongly the fact that in many contexts the phase contains much of the essential "information" in a signal.

Manuscript received April 14, 1980; revised December 16, 1980. This work was supported in part by the Defense Advance Research Projects Agency monitored by the ONR under Contract N00014-75-C-0951-NR049-328 at RLE, in part by the National Science Foundation under Grant ECS8007102, and in part by the Department of the Air Force under Contract F19628-80-C-0002 at M.I.T. Lincoln Laboratory.

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The above discussion relates to the fact that if the true magnitude information is eliminated many of the important characteristics of the signal are nevertheless retained. In the experiments outlined above, the true magnitude information is simply replaced by a standard magnitude. With so much intelligibility incorporated in the phase, it is natural to consider the possibility of *recovering* some or perhaps all of the magnitude information from the phase. It is well known that this is possible under certain assumptions, such as when the signal is minimum phase. Under this assumption, the Hilbert transform can be used to recover the spectral magnitude to within a gain factor from the phase. However, many signals of practical importance are not minimum-phase signals and consequently this procedure has limited applicability. However, as we describe in Section IV, there are other conditions which can be imposed on a signal such that it is *exactly* recoverable to within a scale factor from the phase. As we show, one such condition, which applies to discrete-time signals, is that the signal be of finite duration and have no zero-phase components. This set of conditions applies to a relatively broad class of signals and provides the potential for more precise synthesis of signals from phase information alone.

Sections II, III, and IV demonstrate the importance of phase both empirically and analytically. In Section V, we consider specific algorithmic procedures for reconstructing a signal from phase information alone. This includes combining the phase with a standard magnitude, combining the phase with an estimate of the magnitude, and recovering all or some of the magnitude information from the phase using the theory outlined in Section IV.

The importance of phase in signal representation has a number of important implications with regard to applications. In Section VI, we review several that have been developed or proposed.

II. Phase-Only Fourier Synthesis

Apparently independently, and in a number of different contexts, it has been recognized that many features of a signal are retained in a phase-only Fourier synthesis but not in a magnitude-only Fourier synthesis. Specifically, let $f(\boldsymbol{x})$ denote an n-dimensional signal and $F(\boldsymbol{\omega}) = |F(\boldsymbol{\omega})| e^{j\theta(\boldsymbol{\omega})}$ its n-dimensional Fourier transform where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is the vector of independent variables $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_n)$ is the vector of frequency variables, and $|F(\boldsymbol{\omega})|$ and $\theta(\boldsymbol{\omega})$ are the magnitude and phase, respectively, of $F(\boldsymbol{\omega})$. The magnitude-only Fourier synthesis $f_m(\boldsymbol{x})$ is defined as the signal with Fourier transform $|F(\boldsymbol{\omega})|$, i.e.,

$$\mathcal{F}\{f_m(\boldsymbol{x})\} = |F(\boldsymbol{\omega})|. \tag{1}$$

Correspondingly, the phase-only synthesis $f_p(\boldsymbol{x})$ is to have the Fourier transform

$$\mathcal{F}\{f_p(\boldsymbol{x})\} = M(\boldsymbol{\omega})\, e^{j\theta(\boldsymbol{\omega})} \tag{2}$$

where $M(\boldsymbol{\omega})$ is either unity or perhaps more generally a magnitude function which is in some way representative of the *class* of signals but not obtained from any knowledge of the *specific* signal $f(\boldsymbol{x})$.

Apparently, the first context in which the similarity between a signal $f(\boldsymbol{x})$ and its phase-only synthesis $f_p(\boldsymbol{x})$ had been recognized and demonstrated was in the Fourier synthesis of crystallographic structures [1]–[4]. Typically, in X-ray crystallography, a crystal structure is deduced from the magnitude of the Fourier transform of the structure, as represented by X-ray diffraction data. A commonly used test for the correctness of the deduced crystal structure is to perform a Fourier synthesis using the observed structure amplitudes and the calculated phases associated with the deduced structure and to verify that this diagram gives peaks of the correct magnitudes at the assumed positions of atoms and none elsewhere. Motivated by this procedure, in 1961 Srinivasan [3] reported the results of a number of empirical tests carried out to assess the relative importance of the phase angles and structure amplitudes in a Fourier synthesis. The results of these tests are illustrated in Fig. 1. In Fig. 1(a) is shown a contour diagram of a specific projection of L-tyrosine HCL using correct Fourier transform magnitude *and* phase information. The contours correspond to constant electron density with peaks occurring at the atomic positions. In Fig. 1(b) is shown the electron density contour plot for the same projection obtained by using the same phase information used for Fig. 1(a) but with a magnitude which is constant multiplied by a tapering so that it gradually falls off to zero at the limit of the observed data in the frequency domain. Overlaid in solid lines is the correct atomic structure. It is evident that for the most part the locations and in some cases even the relative strengths of some of the atoms have been synthesized correctly. To further emphasize the importance of the phase information relative to the Fourier magnitudes, Fig. 1(c) shows the synthesis in which the correct phase is combined with Fourier magnitudes which are a random permutation of the true magnitudes. In Fig. 1(d), the Fourier synthesis was obtained using the phase associated with the same structure synthesized in Fig. 1(a) and with a transform magnitude associated with a totally different structure. The locations of the atoms in the structure associated with the transform magnitude are indicated by crosses. Clearly, the synthesized structure in Fig. 1(d) more closely resembles in atomic positions, the structure associated with the phase used in the synthesis, as compared with the structure associated with the magnitude.

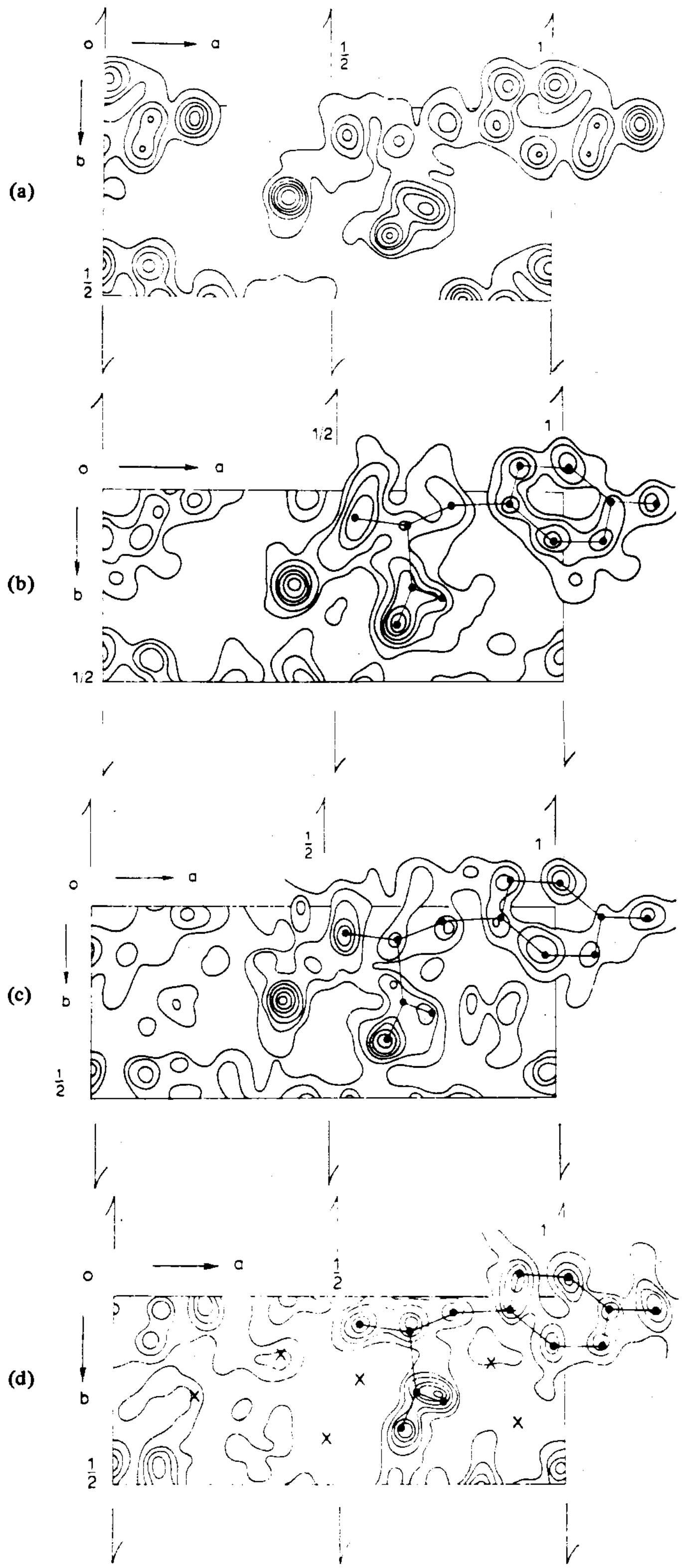


Fig. 1. (a) Contour diagram of a projection of L-tyrosine HCL synthesized from correct Fourier transform magnitude and correct phase. The contours correspond to constant electron density with peaks occurring at the atomic positions. (b) Same as (a), but synthesized from correct phase and a tapered constant magnitude. (c) Same as (a), but synthesized from correct phase and the magnitude which is a random permutation of the true magnitude. (d) Same as (a), but synthesized from correct phase and a magnitude associated with a totally different structure whose locations of the atoms are indicated by crosses (after Ramachandran and Srinivasan [2]).

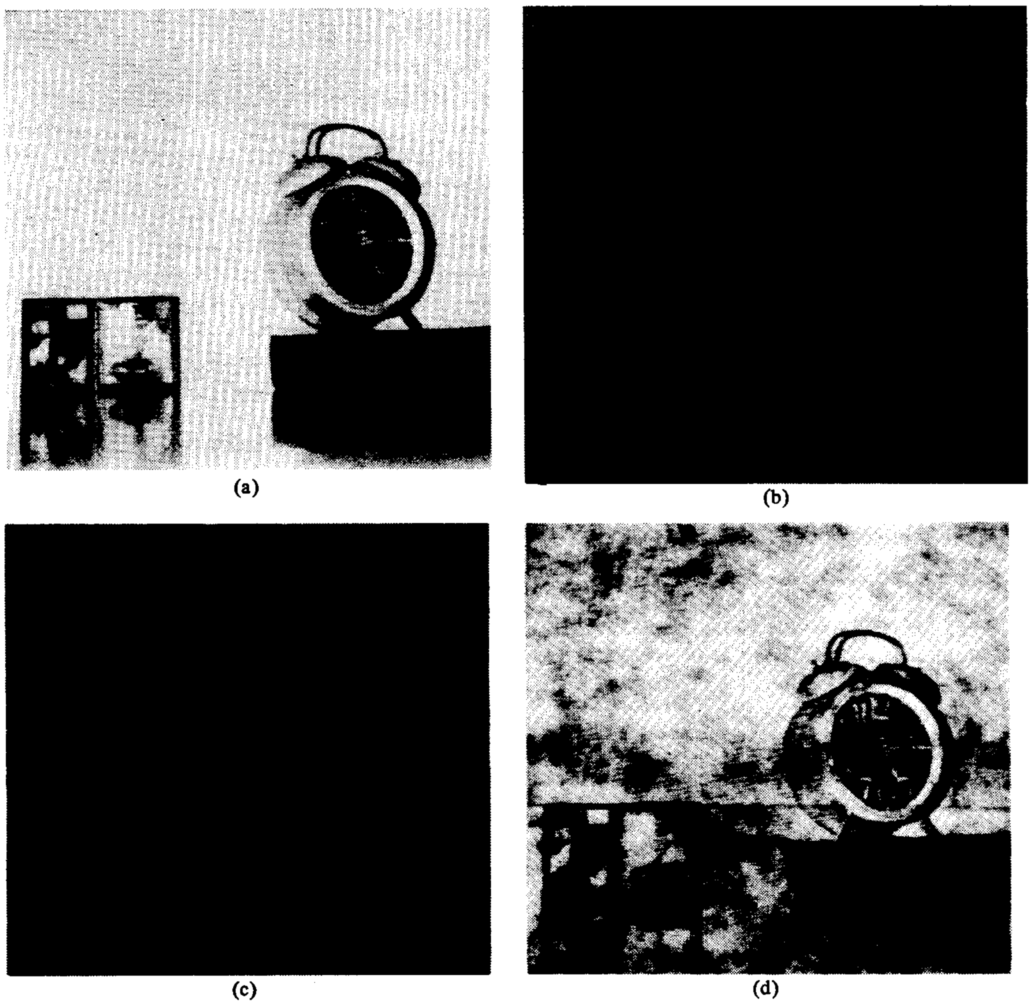


Fig. 2. (a) Original image. (b) Image synthesized from the Fourier transform magnitude of (a) and zero phase. (c) Image synthesized from the Fourier transform phase of (a) and unity magnitude. (d) Image synthesized from the Fourier transform phase of (a) and a magnitude averaged over an ensemble of images.

Closely related to the above experiments but apparently carried out independently is a set of similar experiments in which phase-only and magnitude-only images were compared with an original [5], [6]. The results of such an experiment are illustrated in Fig. 2. Fig. 2(a) corresponds to the original image and thus contains both spectral magnitude and phase. In Fig. 2(b) is shown the magnitude-only image, and in Fig. 2(c) the phase-only image with unity magnitude, i.e., with $M(\boldsymbol{\omega})$ in (2) equal to unity. Fig. 2(d) shows the phase-only image with a spectral magnitude which is the average spectral magnitude over an ensemble of images. From comparison of Figs. 2(a)-(d), it is clear that the phase-only image, but not the magnitude-only image retains many of the features of the original. Similar to Fig. 2(d) and as a further illustration of the relative importance of phase versus magnitude in synthesizing an image, Fig. 3(c) shows the synthesis of an image using the Fourier transform phase associated with the image of Fig. 3(a) and the Fourier transform magnitude associated with the image of Fig. 3(b). Fig. 3(d) shows the synthesis using the Fourier transform phase of Fig. 3(b) and the Fourier transform magnitude of Fig. 3(a). Clearly, in both cases, the reconstructed image most closely resembles the one with the same phase.

A similar set of experiments have been carried out with speech, with similar results. Fig. 4(a) is the spectrogram of a sentence and Fig. 4(b) is the spectrogram of the magnitude-only equivalent, obtained by computing the Fourier transform of the entire sentence and inverse Fourier transforming after setting the phase to zero. In Fig. 4(c) is the spectrogram of the phase-only equivalent for which the true phase is retained and the Fourier transform magnitude is unity. As suggested by the spectrograms and confirmed by listening, intelligibility is lost in the magnitude-only reconstruction but not in the phase-only reconstruction. Analogous to the images in Fig. 3, we show in Figs. 5(a) and (b) the spectrograms of two original sentences. Fig. 5(c) is the spectrogram for the sentence resulting from combining the phase of the sentence in Fig. 5(a) with the magnitude of the sentence in Figs. 5(b) and (d) is the spectrogram for the sentence resulting from combining the phase of the sentence in Fig. 5(b) with the magnitude of the sentence in Fig. 5(a). As with images, the reconstructed speech most closely resembles the one with the same phase.

Another context in which the potential importance of phase-only Fourier synthesis has been recognized is in both acoustical and optical holography [7]-[19]. In both cases, the hologram

Fig. 3. (a) Original image A. (b) Original image B. (c) Image synthesized from the Fourier transform phase of image A and the magnitude of image B. (d) Image synthesized from the Fourier transform magnitude of image A and the phase of image B.

corresponds to the diffraction pattern at a reference plane due to the illumination of an object by a monochromatic source. For a two-dimensional object for example, if the reference plane is sufficiently far from the object, the Fraunhofer approximation can be made and the spatial diffraction pattern $U(u, v)$ is approximately [20]

$$U(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \exp\left[-j \frac{2\pi}{\lambda z}(ux + vy)\right] dx\, dy \tag{3}$$

where $f(x, y)$ is the light or sound amplitude distribution at the two-dimensional object, x and y are the spatial coordinates at the object, u and v are the spatial coordinates at the reference plane, λ is the wavelength of the source and z is the distance from the object to the reference plane. From (3), we recognize the diffraction pattern as the two-dimensional Fourier transform of the object with $2\pi u/\lambda z$ and $2\pi v/\lambda z$ representing the spatial frequency variables ω_x and ω_y, respectively. For the reference plane closer to the object so that the Fresnel approximation to the diffraction pattern is more appropriate, the spatial diffraction pattern is approximately

$$U(u,v) = \frac{e^{jkz}}{j\lambda z} \exp\left[j\frac{k}{2z}(u^2+v^2)\right] \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot \exp\left[j\frac{k}{2z}(x^2+y^2)\right] \cdot \exp\left[-j\frac{2\pi}{\lambda z}(ux+vy)\right] dx\, dy. \tag{4}$$

In this case, the double integral is recognized as the two-dimensional Fourier transform of $f(x,y) \cdot \exp[j(k/2z)(x^2+y^2)]$ and thus the Fresnel diffraction pattern is the Fourier transform multiplied by a known phase factor. In both the acoustical case and the optical case, the possibility of reconstructing the object from only the phase of the diffraction pattern has been proposed and investigated and in particular it has been demonstrated that reasonable representations of the object can

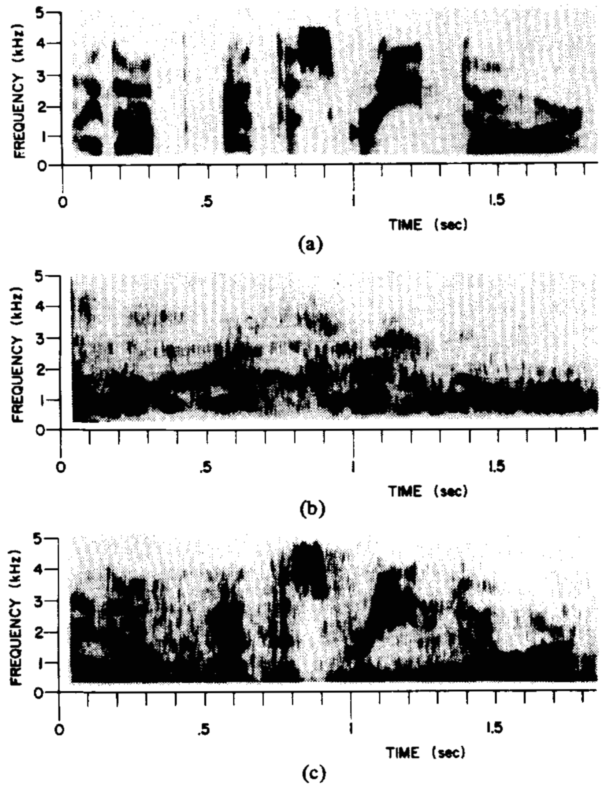


Fig. 4. (a) Spectrogram of an original sentence "Line up at the screen door." (b) Spectrogram obtained from the Fourier transform magnitude of the entire sentence in (a) and zero phase. (c) Spectrogram obtained from the Fourier transform phase of the entire sentence in (a) and unity magnitude.

be obtained using this phase-only synthesis. In the optical case, such holograms have been referred to as kinoforms. They appear to have potential importance for computer generated holograms since they suggest that in some cases, only the phase of the diffraction pattern need be synthesized.

The above examples serve to demonstrate that in a number of contexts, the Fourier transform phase contains more of the "important" information than the Fourier transform magnitude, and that the phase-only synthesis has a high degree of "intelligibility." Clearly, elimination of the true spectral magnitude destroys some aspects of the signal, and the implication is that these are not as important in the implied task for which the signal is to be used. As summarized in the next section, there have been a number of attempts to justify both heuristically and analytically the observation that the spectral phase appears to be more important than the spectral magnitude. While none of the individual justifications can be considered to be general and totally conclusive they all, in different ways, support the general observation.

III. Some Justification for the Importance of Phase

Several approaches to justifying the relative importance of phase have been based on statistical arguments. For example, Tescher [21] has considered the rms error due to spectral phase and amplitude quantization for random signals and concluded that quantization of the phase requires approximately two more bits than quantization of the amplitude for the same rms error. A similar conclusion was reported by Pearlman and Gray [22]. In their analysis, distortion rate theory is applied to real-part, imaginary-part, and magnitude-phase encoding of the discrete Fourier transform of random sequences, and they conclude that for equivalent distortion, the phase angle must be encoded with 1.37 bits more than the magnitude.

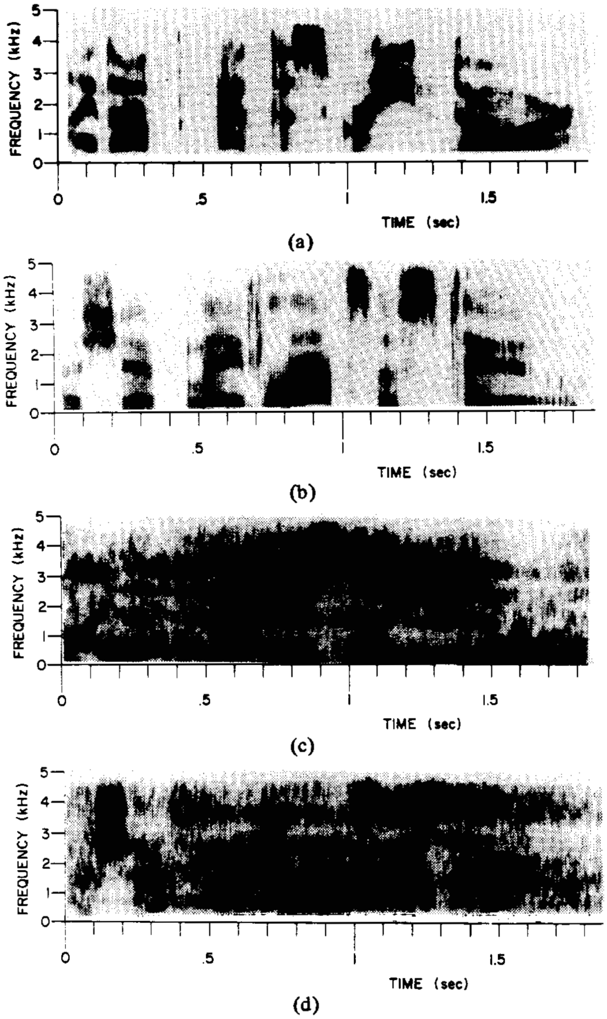


Fig. 5. (a) Spectrogram of original sentence A. (b) Spectrogram of original sentence B. (c) Spectrogram obtained from the Fourier transform phase of sentence A and the magnitude of sentence B. (d) Spectrogram obtained from the Fourier transform phase of sentence B and the magnitude of sentence A.

From a very different point of view, Kermisch [10] has reached a similar conclusion based on an analysis of image reconstruction from kinoforms. In his analysis, he develops an expansion of the phase-only reconstructed image $I(x, y)$ in the form

$$I(x,y) = A\,[I_0'(x,y) + (1/8)\,I_0'(x,y) * R_0'(x,y) + (3/64)\,I_0'(x,y) * R_0'(x,y) * R_0'(x,y) + \cdots] \quad (5)$$

where $I_0'(x, y)$ is the normalized irradiance of the original object, $R_0'(x, y)$ is the two-dimensional autocorrelation function of $I_0'(x, y)$ and $*$ denotes the two-dimensional convolution operator. The second and higher order terms in (5) represent the degradation. Integrating (5) and using the fact that $I_0'(x, y)$ has been normalized to have unit area, Kermisch concludes that

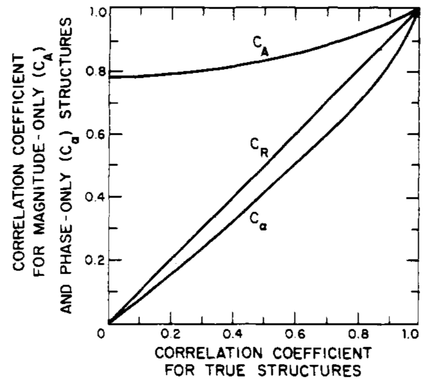


Fig. 6. Comparison of correlation coefficients for true structures, magnitude-only structures, and phase-only structures (after [4]).

the first term $I_0'(x, y)$ contributes approximately 78 percent to the total radiance in the image plane and the higher order terms approximately 22 percent, or a ratio of approximately 1.8 bits.

Srinivasan and Chandrasekaran [4] approached the explanation of the "intelligibility" of phase-only synthesis in a slightly different way. In the context of their application, the reconstruction of atomic structures, they carried out a statistical analysis, relating the correlation of two structures to the correlation of their magnitude-only reconstructions and phase-only reconstructions. Their results for the case of nonsymmetric structures are shown plotted in Fig. 6, where the horizontal axis is the correlation between two structures. The curve C_A represents the correlation between the corresponding magnitude-only reconstructions as a function of the correlation between the true structures. Likewise, the curve C_α represents the correlation between the phase-only reconstructions as a function of the correlation between the true structures. The line C_R then represents the true correlation for the original structures. We note in particular that the correlation C_α between the phase-only reconstructions follows very closely the true correlation C_R whereas the magnitude-only correlation C_A has a much smaller range of variation and is not close to the true correlation.

The fact that phase-only reconstruction preserves much of the correlation between signals would suggest that the "location" of events tends to be preserved. A related observation that can be made about the examples of Figs. 1, 2, and 3 is that much of the intelligibility in the original and reconstructed signals is related to the location of "events" such as lines, points, etc. It is generally plausible that the phase reflects the location of events more than magnitude and, at least for simple examples, this is exactly true. For example, a translation in position (time or space) of a signal has no effect on the Fourier transform magnitude and affects only the phase, in particular, by adding a linear phase term. As a further indication of the plausibility of this statement, consider the effect of filtering a one-dimensional signal with a system whose frequency response has zero phase. The impulse response of such a system tends to be concentrated about the origin and consequently the effect of this on a narrow event in time such as an impulse will be to replace it by the filter impulse response concentrated around the same time as the original event. In fact, if the zero-phase filter contains only zeros in its transfer function, the impulse response will consist of a linear combination of impulses, doublets and higher order singularities, all occurring at $t = 0$. The effect of this on a narrow event in time such as an impulse is to replace it at the same time location by this linear combination of singularity functions. Thus, in fact, zero-phase filtering, which may seriously distort the spectral magnitude will still preserve, and may in fact sharpen narrow time events. Clearly, a similar argument applies for zero-phase filtering in the multidimensional case.

The above informal discussion suggests one interpretation of the importance of phase in relation to the preservation of the location of "events." Another similarly informal justification for the intelligibility of the phase-only reconstructions in Section II derives from the interpretation of the generation of the phase-only signal with unity magnitude as a spectral whitening process. Specifically, with $M(\omega)$ chosen as unity in (2),

$$\mathfrak{F}[f_p(x)] = \frac{1}{|F(\omega)|}\, \mathfrak{F}[f(x)]. \tag{6}$$

Since the spectral magnitude of speech and pictures tends to fall off at high frequencies, the phase-only signal $f_p(x)$ will, among other effects, experience a high-frequency emphasis which will accentuate lines, edges and other narrow events without modifying their position. Although the above argument provides a general basis for an intepretation of the results, it is not precise enough for a complete explanation. The basic processing to obtain the phase-only signal is of course highly nonlinear and the simplified interpretation above assumes that it can be viewed as a linear process. Furthermore, while it is reasonable to identify $1/|F(\omega)|$ as generally emphasizing high frequencies over low frequencies, it will also have specific spectral details associated with it which for some examples could certainly affect or obliterate important features in the original signal. Several simple examples serve to illustrate the point. Consider, for example, any zero-phase or linear-phase signal. Clearly, the phase-only counterpart will consist only of a single impulse with a position dictated by the slope of the linear phase. Thus, for example, as was illustrated in Fig. 4, the phase-only equivalent of speech is highly intelligible. However, we can consider constructing a zero-phase sentence by concatenating this original with itself reversed in time, with an impulse separating them to ensure that the Fourier transform is positive. In this case, the phase-only equivalent signal will contain *only* an impulse. As another illustration, if, instead of a sentence as was the case in Fig. 4, we consider a steady-state vowel, the magnitude of the long-time Fourier transform will again fall off at high frequencies as we would expect but it will also contain the resonances associated with the formants of the steady-state vowel. Thus the signal obtained from only the phase of the long-time Fourier transform would not be expected to contain the formants of the original vowel and thus the essential features of the original signal will have been lost. This is illustrated in Fig. 7 where Fig. 7(a) is the spectrogram of a steady-state vowel and Fig. 7(b) is the spectrogram of the phase-only reconstruction.

The above examples suggest that there are factors to consider in addition to the basic whitening property of the phase-only reconstruction. In particular, we propose that for both speech and pictures, if the long-time Fourier transform is "sufficiently smooth," then intelligibility will be retained in the phase-only reconstruction. This condition can be interpreted in several ways. For speech, if the long-time transform is relatively smooth, the essential formant structure of the short-time transform will remain intact in the whitening process. For the

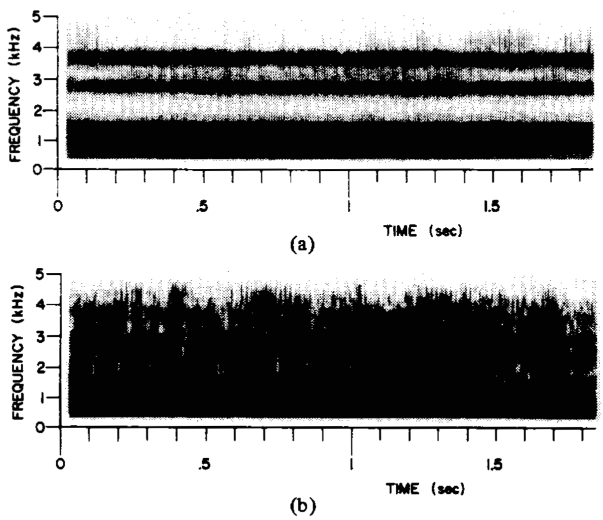


Fig. 7. (a) Spectrogram of a steady-state vowel. (b) Spectrogram obtained from the Fourier transform phase of the steady-state vowel in (a) and unity magnitude.

case of images, if the long-time transform magnitude is smooth and falls off at high frequencies then the principal effect of the whitening process is to emphasize the high frequencies and therefore the edges in the image, thereby retaining many of the recognizable features. Smoothness of the long-time transform magnitude equivalently implies a short impulse response for the whitening filter, and a short correlation function for the original signal. Although shortness of the correlation function does not by itself guarantee a smooth transform magnitude, we believe that it is an important factor.

Importance of the long-time phase does not, of course, imply importance of short-time phase. This is certainly evident in the case of speech where it is well known that on a short-time basis, phase is relatively unimportant [23], [24] whereas, as evidenced in our discussion above, on a long-time basis, the signal constructed from only the phase is intelligible and retains many of the important features of the original.

The difference in importance of phase on a long-time versus short-time basis has consequences in a variety of practical areas including filtering and transform coding. In filtering, the importance of the phase of the filters is, of course, associated with the importance of phase on the time (or space) scale commensurate with the length of the filter impulse response. Thus, for example, the fact that for many pictures, the phase of the overall image is important does not by itself imply that in image filtering, particular attention must be paid to the phase characteristics of the filter. In transform coding, the difference in importance of magnitude and phase as the size of the transform increases clearly indicates that the relative importance of accurate coding of each of these components changes.

IV. Exact Representation of a Signal by Its Phase

In Section II, we considered the Fourier synthesis of signals using correct phase information and a spectral magnitude which is perhaps representative of the class of signals but is otherwise unrelated to the specific signal to be synthesized. The reasonably high "intelligibility" of the results demonstrates the fact that much of the important information resides in the phase and raises the question as to whether some or perhaps all of the magnitude information can be extracted or inferred from the phase.

In general, a signal is not uniquely defined by its phase, as is illustrated by the observation that a signal convolved with any zero-phase signal will produce another signal with the same phase. Thus, without some assumptions about the signal, the phase may, at best, uniquely specify a signal only to within an arbitrary zero-phase factor. However, if some additional knowledge is available, then under certain conditions a signal may be uniquely defined by its phase. One well-known set of conditions under which a signal may be uniquely recovered to within a scale factor from its phase is the minimum-phase or maximum-phase condition. For continuous-time signals with rational Laplace transforms, this condition corresponds to requiring that all the poles and zeros lie only in the left half or only in the right half of the s-plane. For discrete-time signals with rational z-transforms, the corresponding condition is that all poles and zeros lie only inside or only outside the unit circle in the z-plane. Under these conditions, the log magnitude of the Fourier transform is the Hilbert transform of the phase.

For many signals of interest, including those illustrated in Section II, the minimum-phase or maximum-phase condition does not generally apply. There are, however, other sets of conditions unrelated to the minimum-phase or maximum-phase conditions under which a signal is recoverable (again, to within a scale factor) from its phase. In particular, we summarize below a number of statements relating to conditions which would seem to be satisfied by a broad class of signals, under which one-dimensional and multidimensional discrete-time signals can be exactly recovered from their phase [25]. The proofs of these statements can be found in the related references.

Theorem 1: A one-dimensional sequence which is finite in length and has a z-transform with no zeros on the unit circle and no zeros in conjugate reciprocal pairs is uniquely specified to within a scale factor by the phase of its Fourier transform (or by the tangent of its phase).

The condition which excludes zeros from the unit circle is made only for convenience. The condition which excludes zeros in conjugate reciprocal pairs, however, is necessary to eliminate the possible ambiguity due to zero-phase components. This theorem can be modified to be applicable to all-pole sequences since the convolutional inverses of these sequences are finite in length.

Although Theorem 1 is formally stated for one-dimensional sequences, an extension to n-dimensional sequences has been accomplished by using the projection-slice theorem. This theorem establishes the result that an n-dimensional sequence having a rational z-transform may be mapped into a one-dimensional sequence (projection) by means of an invertible transformation [26]. This transformation has the important property that the phase of the projection is uniquely defined by the phase of the n-dimensional sequence; specifically, the phase of the projection is equal to a slice of the phase of the n-dimensional sequence. Consequently, the multidimensional phase-only problem can be mapped into a one-dimensional phase-only problem and the phase-only reconstruction theorem for one-dimensional sequences may be used.

The approach of transforming n-dimensional sequences into one-dimensional projections provides one basis for extending Theorem 1 to n-dimensional signals. However, this approach imposes constraints on a projection of the n-dimensional sequence rather than directly on the n-dimensional sequence. To impose the constraints directly on the n-dimensional sequence, a more general theorem that reduces to Theorem 1 for one-dimensional sequences has been developed. In essence, this Theorem states that an n-dimensional sequence which is finite in extent and which has an n-dimensional z-transform with no symmetric factors is uniquely specified by its n-dimensional Fourier transform phase. The proof of this more general theorem is less straightforward than that required in the one-dimensional case due to the absence of a Fundamental Theorem of Algebra. The specific statement and proof of this more general theorem can be found in [27].

Although the phase-only reconstruction theorems specify a set of conditions under which a sequence is uniquely specified to within a scale factor by its phase, it is assumed that the phase is known for all frequencies. Since any practical algorithm for reconstructing a sequence from its phase will base the reconstruction on only a finite set of samples of the phase, the one-dimensional theorem has been extended to consider the uniqueness of a sequence based only on samples of its phase. Specifically,

Theorem 2: A sequence which is known to be zero outside the interval $0 \leqslant n \leqslant (N-1)$ is uniquely specified to within a scale factor by $(N-1)$ distinct samples of its phase (or tangent of its phase) in the interval $0 < \omega < \pi$ if it has a z-transform with no zeros on the unit circle or in conjugate reciprocal pairs.

Even though Theorem 2 cannot be extended directly to n-dimensional sequences, it has been shown [27] that an n-dimensional sequence which has a finite extent of $N_1 \times N_2 \times \cdots \times N_n$ points and which has an n-dimensional z-transform with no symmetric factors is uniquely specified to within a scale factor by the phase of its n-dimensional DFT with DFT size $2N_1 \times 2N_2 \times \cdots \times 2N_n$. This result forms the basis for developing the signal reconstruction algorithms which are described in Section V. Unlike Theorem 2 for one-dimensional signals, this result requires more independent phase samples than the size of the signal and the phase samples have to be taken on a rectangular grid. Work to generalize this result so that Theorem 2 becomes a special case when $n = 1$ is currently in progress.

V. Algorithms for Image Reconstruction from Phase

In the discussion in the previous section, we have seen that a signal can often be recovered completely or in part from knowledge of its phase alone. Depending on what is known about the signal in addition to the given phase function and how this additional knowledge is specifically exploited in the signal reconstruction, a variety of different algorithms can be developed. In this section, we consider a number of algorithms for the reconstruction. While all of the algorithms apply in the general context of n-dimensional signals, we will phrase the discussion and illustrate the procedures specifically in the context of images, and consider a number of approaches to image reconstruction from the Fourier transform phase when no information or only partial information is available about the Fourier transform magnitude. We begin with reconstruction procedures based on the discussion in Section II, whereby the phase is combined with a "representative" magnitude, or with additional available information about the magnitude, but no attempt is actually made to recover correct magnitude information from the phase. We then consider algorithms for the exact reconstruction of signals from phase information alone, based on the theory outlined in Section IV.

Based on the discussion in Section II, one approach to image reconstruction from its phase function is the phase-only synthesis of (2). As was illustrated in Fig. 2, the phase-only synthesis with $M(\boldsymbol{\omega})$ taken as unity preserves many of the important features of the original image such as the edge information. Even though image reconstruction based on the unity spectral magnitude assumption is simple and preserves many important features of the original image, the unity magnitude assumption is quite arbitrary and the overall quality of images reconstructed in this way tends to have the appearance of a broad-band noise background. An alternative approach to image reconstruction from its phase function is to assume a spectral magnitude $M(\boldsymbol{\omega})$ which is more consistent with that of a typical image. For example, for a typical image, the spectral magnitude generally decreases as the frequency increases. These spectral characteristics of images can be incorporated in the image reconstruction algorithm by assuming the spectral magnitude to be the average magnitude over typical images or prototype images with similar contents and then reconstructing an image by combining this average magnitude with the given phase function. This was illustrated in Fig. 2(d) where the spectral magnitude used was obtained by averaging the spectral magnitudes of five different images whose contents have little relation with the original picture in Fig. 2(a). Clearly, the image in Fig. 2(d) has a high degree of intelligibility and appears to be somewhat more natural than the image obtained from the unity magnitude assumption.

The image reconstruction algorithms discussed above do not require any further knowledge about the image beyond the given phase function. As will be discussed in Section VI, in some applications such as blind deconvolution, some information about the spectral magnitude of the image may be available in a degraded form. For this class of problems, this additional information about the image may be incorporated in developing an algorithm to reconstruct an image from its phase function [28], [30]. For example, suppose the degraded spectral magnitude $M_y(\boldsymbol{\omega})$ available for image reconstruction can be represented by

$$M_y(\boldsymbol{\omega}) = M_f(\boldsymbol{\omega}) \cdot M_b(\boldsymbol{\omega}) \tag{7}$$

where $M_f(\boldsymbol{\omega})$ represents the spectral magnitude of the original image $f(x)$ and $M_b(\boldsymbol{\omega})$ represents the degradation and has the property that it is a smooth magnitude function. To reconstruct an image from the given phase function and $M_y(\boldsymbol{\omega})$, we may attempt to first estimate $M_f(\boldsymbol{\omega})$ from $M_y(\boldsymbol{\omega})$ and then combine it with the phase function. To estimate $M_f(\boldsymbol{\omega})$ from $M_y(\boldsymbol{\omega})$, $M_y(\boldsymbol{\omega})$ may be smoothed with a smoothing operator "S" so that from (7)

$$S\{M_y(\boldsymbol{\omega})\} = S\{M_f(\boldsymbol{\omega}) \cdot M_b(\boldsymbol{\omega})\}. \tag{8}$$

Since $M_b(\boldsymbol{\omega})$ is a smooth function, $S\{M_f(\boldsymbol{\omega}) \cdot M_b(\boldsymbol{\omega})\}$ may be approximated by $S[M_f(\boldsymbol{\omega})] \cdot M_b(\boldsymbol{\omega})$, and therefore

$$S\{M_y(\boldsymbol{\omega})\} \simeq S[M_f(\boldsymbol{\omega})] \cdot M_b(\boldsymbol{\omega}). \tag{9}$$

Combining (7) and (9),

$$M_f(\boldsymbol{\omega}) \simeq M_y(\boldsymbol{\omega}) \cdot \frac{S[M_f(\boldsymbol{\omega})]}{S[M_y(\boldsymbol{\omega})]}. \tag{10}$$

Assuming that the smoothed form of $M_f(\boldsymbol{\omega})$ is approximately equal to the smooth form of the average spectral magnitude

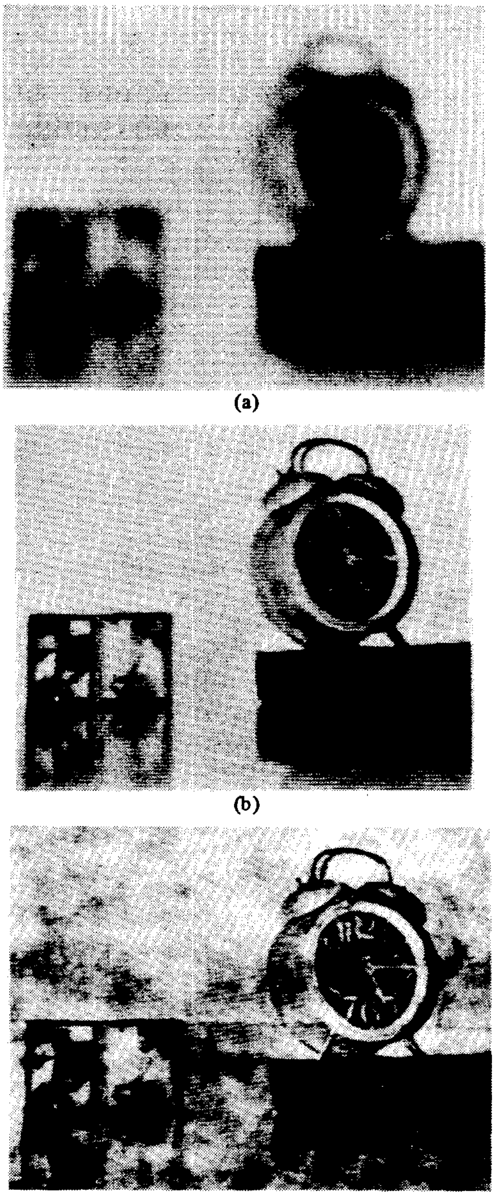


Fig. 8. (a) The image in Fig. 2(a) blurred by a Gaussian shaped point spread function. (b) Image reconstructed from the Fourier transform phase of the image in (a) and a magnitude estimated from the degraded magnitude. (c) Image reconstructed from the Fourier transform phase of the image in (a) and a magnitude averaged over an ensemble of images.

over typical images, from (10), the estimate $\hat{M}_f(\boldsymbol{\omega})$ of $M_f(\boldsymbol{\omega})$ is taken as

$$\hat{M}_f(\boldsymbol{\omega}) = M_y(\boldsymbol{\omega}) \cdot \frac{S[M_{av}(\boldsymbol{\omega})]}{S[M_y(\boldsymbol{\omega})]}. \tag{11}$$

Equation (11) can be used as a basis for estimating $M_f(\boldsymbol{\omega})$ from $M_y(\boldsymbol{\omega})$. Once $M_f(\boldsymbol{\omega})$ is estimated, the estimate $M_f(\boldsymbol{\omega})$ can be combined with the given phase function $\theta_f(\boldsymbol{\omega})$ so that the Fourier transform of the reconstructed image $\hat{f}(x)$ is

$$\mathcal{F}\{\hat{f}(x)\} = \hat{M}_f(\boldsymbol{\omega})e^{j\theta_f(\boldsymbol{\omega})}. \tag{12}$$

As an illustration, in Fig. 8(a) is shown an image for which the degradation function $M_b(\boldsymbol{\omega})$ is a Gaussian shaped function. In

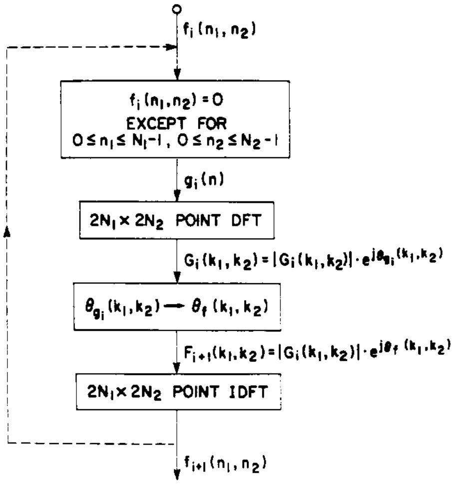


Fig. 9. Iterative algorithm for the finite-extent image reconstruction from its DFT phase samples.

Fig. 8(b) is the reconstruction based on (11) and (12). In Fig. 8(c) is shown an image which results when an average magnitude rather than the estimate based on (10) is combined with the given phase function.

In the image reconstruction algorithms discussed above, no attempt is made to recover spectral magnitude information from the phase function. As discussed in Section IV, there exist conditions under which an image or equivalently the spectral magnitude is uniquely specified to within a scaling factor by its phase function. For the class of images that satisfy this condition, there exist two numerical algorithms that may be used for the reconstruction. In summarizing the numerical algorithms, $f(n_1, n_2)$ is used to denote a two-dimensional image and $\theta_f(\omega_1, \omega_2)$ denotes the phase of $f(n_1, n_2)$. The sequence $f(n_1, n_2)$ is assumed to have no symmetric factors in its z-transform, and to be zero outside the interval $0 \leqslant n_1 \leqslant N_1 - 1$, $0 \leqslant n_2 \leqslant N_2 - 1$, so that the conditions for the unique specification by phase are satisfied. It is further assumed that $f(0, 0)$ is not zero and is known so that the reconstructed image can be properly scaled.

The first algorithm is an iterative technique [31] which is in a form similar to the Gerchberg–Saxton algorithm [32] and several iterative algorithms developed by Quatieri [33]. This method involves repeated transformation between the time and frequency domains with the known constraints imposed in each domain. Thus, at the ith iteration, the current estimate of the sequence is Fourier transformed and the resulting phase is replaced with the given phase. Inverse Fourier transforming, the $(i + 1)$th estimate is formed by setting the points outside the interval $0 \leqslant n_1 \leqslant N_1 - 1$, $0 \leqslant n_2 \leqslant N_2 - 1$ equal to zero. This iterative algorithm is summarized in Fig. 9 using the DFT and inverse DFT (IDFT) operations. Convergence of the algorithm has been shown theoretically [34] and observed empirically. A number of acceleration techniques have also been developed which considerably improve the rate of convergence of the iterative procedure [35]. An example of the iterative algorithm in which an image of 128 X 128 pixels is reconstructed from its phase function using an acceleration technique is shown in Fig. 10. The images in the figure correspond to the original, the unity magnitude phase-only reconstruction used as an initial estimate in the iteration,

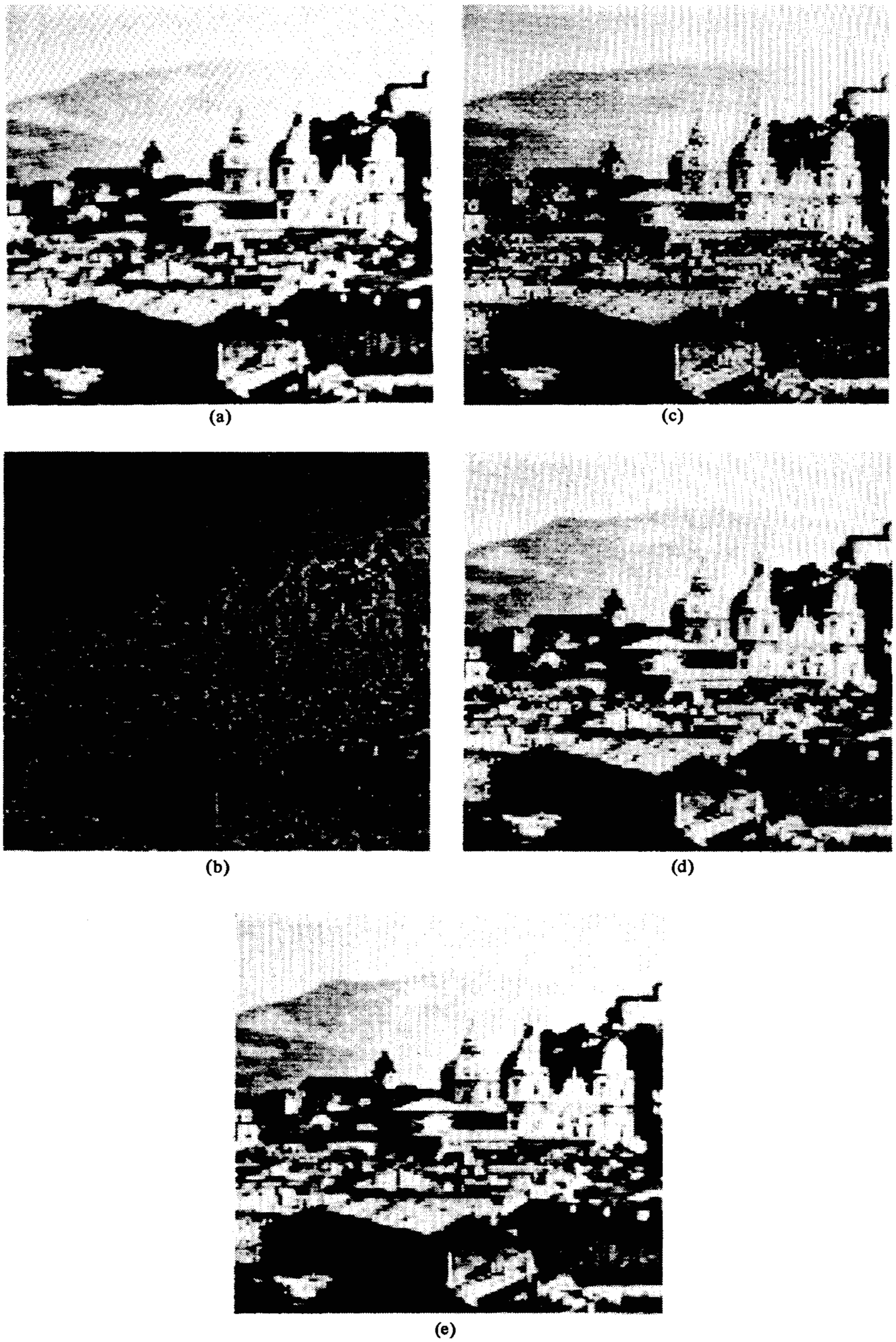


Fig. 10. (a) Original image. (b) Initial image estimate used in the iteration. The estimate was obtained by combining the Fourier transform phase of the image in (a) and unity magnitude. (c) Image reconstructed after 10 iterations. (d) Image reconstructed after 30 iterations. (e) Image reconstructed after 50 iterations.

and the results after 10, 30, and 50 iterations. The result after 50 iterations is visually indistinguishable from the original.

An alternative algorithm for exactly reconstructing an image from its phase function involves solving a set of linear equations and leads to a closed form solution. Representing the phase function of $f(n_1, n_2)$ by $\theta_f(\omega_1, \omega_2)$, it can be shown from the definition of $\theta_f(\omega_1, \omega_2)$ that

$$\sum_{\substack{n_1=0 \\ (n_1,n_2)\neq(0,0)}}^{N_1-1} \sum_{n_2=0}^{N_2-1} f(n_1, n_2) \cdot \{\cos(\omega_1 n_1 + \omega_2 n_2) + \tan \theta_f(\omega_1, \omega_2) \cdot \sin(\omega_1 n_1 + \omega_2 n_2)\} = f(0, 0). \quad (13)$$

When sampled at $4N_1N_2$ equally spaced points in ω corresponding to the $2N_1 \times 2N_2$ point DFT, this equation can be viewed as a set of $4N_1N_2$ linear equations in the $N_1N_2 - 1$ unknown values of $f(n_1, n_2)$. Approximately half of these equations can be eliminated by exploiting the fact that $\theta_f(\omega_1, \omega_2) = \theta_f(-\omega_1, -\omega_2)$ for real signals and it can be shown [27] that given $f(0,0)$, the remaining set of linear equations can be solved to uniquely determine $f(n_1, n_2)$ for $0 \leqslant n_1 \leqslant N_1 - 1$, $0 \leqslant n_2 \leqslant N_2 - 1$ using generalized inverses and this unique solution is the desired one. For one-dimensional signals, sampling at any $N - 1$ distinct frequencies in the interval $0 < \omega < \pi$ in the one-dimensional form of (13) leads to a set of $N - 1$ linear equations for $N - 1$ unknowns and it can be shown [25] that the set of $N - 1$ linear equations can be solved to uniquely obtain the desired result. An example of the closed form solution in which an image of 16×16 pixels is reconstructed from its phase function is shown in Fig. 11. The images in the figure have been expanded for visual purposes by a zeroth-order hold and the reconstructed image is indistinguishable from the original.

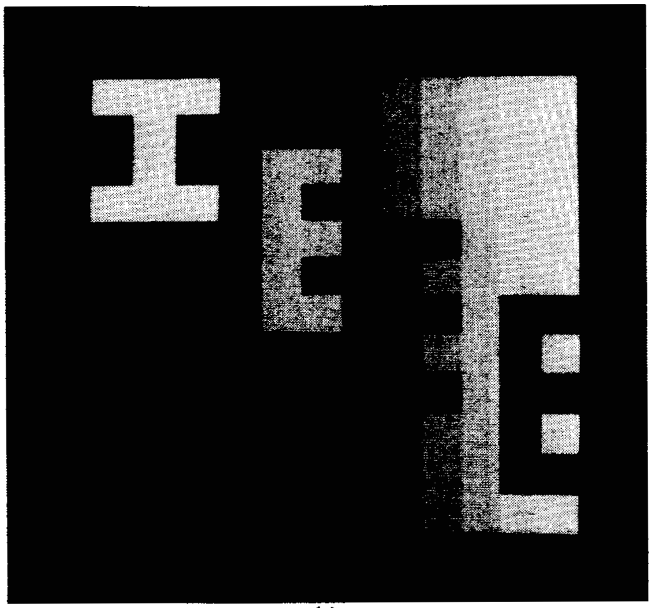
Even though the two algorithms discussed above may be used in principle to recover an image from its phase function, their application in practice is limited due to the computational complexity and their potential sensitivity to inaccuracies in the given phase function. Specifically, the iterative technique requires two $M_1 \times M_2$ point FFT's in each iteration where M_1 and M_2 are 2^9 for an image of 256×256 pixels and many iterations may be required to reach a convergent solution. The closed form solution technique requires solving a set of approximately 2^{16} linear equations for an image of 256×256 pixels. Furthermore, in practice, the *exact* phase function of the image cannot be expected to be available. For the first three algorithms discussed in this section, inaccuracy in the given phase function does not affect the spectral magnitude used in the image reconstruction. However, the above two algorithms capable of exact image reconstruction obtain the spectral magnitude from the given phase function and thus such algorithms tend to be more sensitive to inaccuracies in the phase.

In this section, we have discussed various different ways to reconstruct an image from its phase function. Even though the algorithms discussed are by no means exhaustive, they illustrate that an intelligible image may be recovered from its phase function depending on the specific context in which the image reconstruction problem arises. In the next section, we discuss a variety of different image processing problems in which the problem of recovering an image from its phase function potentially arises.

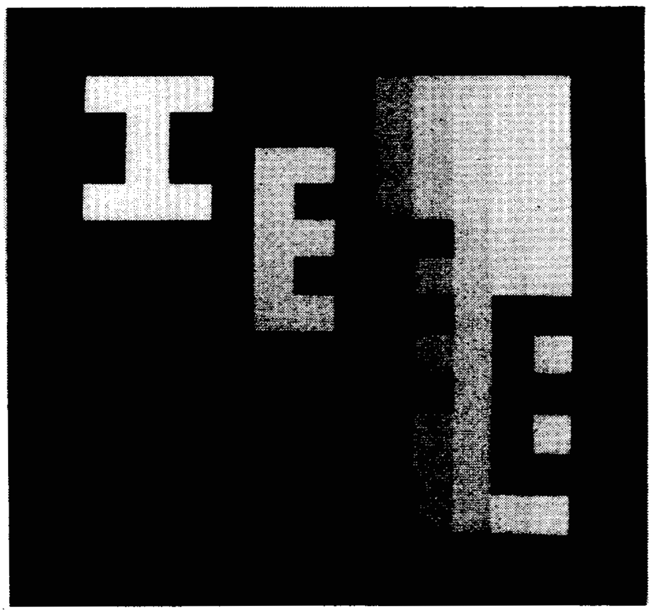
VI. Applications

There are a number of practical image processing problems in which the importance of phase in images and phase-only image reconstruction algorithms have been exploited or can potentially have an impact. For example, methods have been proposed for using phase-only images for image alignment [37], essentially taking advantage of the fact that the autocorrelation function for phase-only signals will always be an impulse.

Another such problem frequently encountered in image processing is that of blind deconvolution whereby an image has been degraded by a blurring function about which detailed knowledge is not available, as for example, when an image is blurred by an optical system whose transfer function is not known. Since in blind deconvolution little is known about either the desired signal or the distorting signal, this problem has been quite difficult and proposed techniques have met with varied success [28]-[30]. In some special cases, however, such as when images are blurred by defocused lenses with circular aperture stops or by long-term exposure to atmospheric turbulence, the distorting signal is known to have a phase function which is approximately zero and consequently the phase of the blurred image is very similar to that of the original image. In such cases, the blind deconvolution problem may be viewed as a problem in which image reconstruction is desired from its phase function and thus the algorithms discussed in Section V may be applicable. Oppenheim *et al.* [6] considered using unity or average spectral magnitude when the blurring function is zero phase. The application of this approach to blind deconvolution in the context



(a)



(b)

Fig. 11. (a) Original image of 16×16 pixels. (b) Reconstructed image from its Fourier transform phase using the closed form solution technique of (13).

of seismic data processing was also explored by Ulrych [36]. Stockham *et al.* [29] developed a homomorphic algorithm for blind deconvolution applicable to cases in which the blurring function is zero phase and very short in its duration. Pohlig *et al.* [30] proposed an algorithm for blind deconvolution utilizing (11) and (12). This class of methods is applicable to cases in which the blurring function is zero phase and its Fourier transform magnitude is a smooth function.

As has been discussed in Section IV, an image can be exactly recovered from its phase function to within a scaling factor if the image satisfies the appropriate conditions. These results indicate that the blind deconvolution problem can in theory be exactly solved to the extent that the blurring function is truly zero and the degradation model is truly convolutional. Despite the computational complexity and potential sensitivity to deviations from the zero phase condition of the blurring function and the convolutional degradation model, in our opinion this approach to the blind deconvolution problem deserves further study.

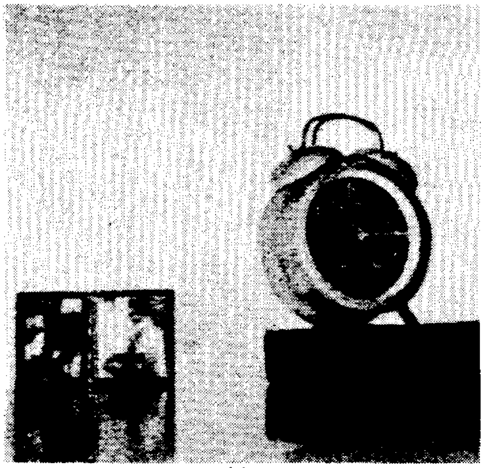
Another area in which the importance of phase in image intelligibility has been recognized is image coding. In image coding by Fourier transform techniques [38], [39], both the phase and magnitude are typically coded and transmitted. In developing coding schemes for the phase and magnitude, it has been found that assigning considerably more bits to coding the phase than the magnitude is important in the success of Fourier transform image coding. In view of our discussions in Section V in which various algorithms to reconstruct an image from its phase function are considered, there appear to be several potential alternative approaches to Fourier transform image coding. For example, since in theory most finite duration sequences, and consequently the spectral magnitude can be recovered to within a scaling factor from the phase function alone, it may be reasonable to attempt to recover some magnitude information from the coded phase, supplemented by magnitude information that was separately coded.

One area which relies heavily on the high intelligibility of phase-only images and in which there appears to be considerable room for further improvement is the kinoform. As we have discussed, the kinoform is a device which records the Fourier transform phase of an image and then reconstructs the image by combining the recorded phase with a constant magnitude. Since the phase of an image is preserved in the kinoform, other methods may be used in reconstructing the image. For example, we may combine the phase with an average magnitude function rather than constant magnitude or attempt to exactly recover the image from the recorded phase. These alternative approaches could potentially improve the performance of the kinoform.

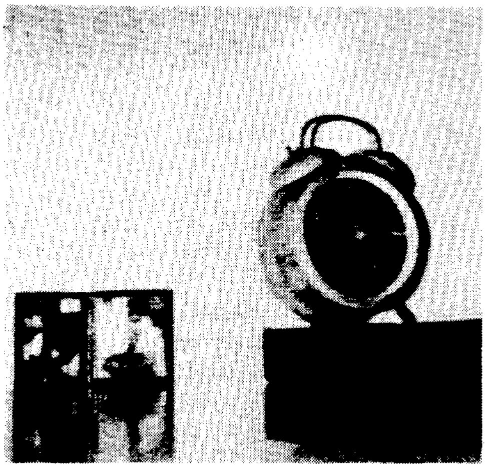
An image processing problem in which an attempt to exploit the importance of phase in images has the potential to have an impact is the restoration of images degraded by additive noise. Examples in which such degradations arise include sensor noise and quantization noise [40] in low-data transmission systems. A common approach [38], [39], [41], [42] to the restoration of images degraded by additive noise is to use a filter whose frequency response $H(\boldsymbol{\omega})$ is given by

$$H(\boldsymbol{\omega}) = \left[\frac{P_f(\boldsymbol{\omega})}{P_f(\boldsymbol{\omega}) + \alpha P_n(\boldsymbol{\omega})}\right] \quad (14)$$

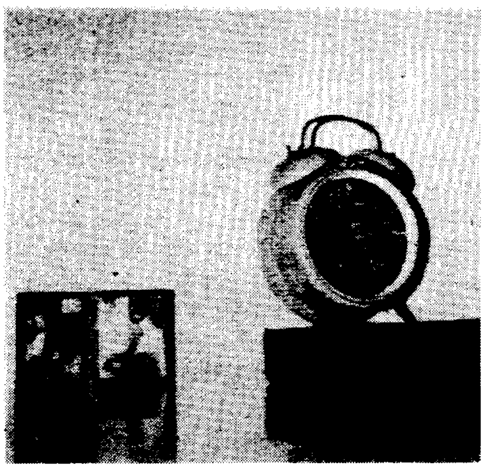
where $P_f(\boldsymbol{\omega})$ and $P_n(\boldsymbol{\omega})$ represent the power spectrum of the image and the additive noise, respectively. For example, equation (14) corresponds to Wiener filtering when $\alpha = \beta = 1$ and power spectrum filtering when $\alpha = 1$ and $\beta = \frac{1}{2}$. From (14), the restoration filter is noncausal with an even, real frequency response and consequently the phase of the filter is zero. Thus the output of the restoration filter, corresponding to the estimate of the original image, has a spectral magnitude which has been modified by the restoration filter, but the phase of the restored image is identical to the phase of the degraded image.



(a)



(b)



(c)

Fig. 12. (a) Image degraded by additive white noise at SNR of 5 dB. (b) Image synthesized from the Fourier transform phase of the noisy image in (a) and the magnitude of the original undegraded image. (c) Image synthesized from the Fourier transform magnitude of the noisy image in (a) and the phase of the original undegraded image.

If we attempt to more accurately estimate the phase as well as the magnitude of the image, there is the possibility that some improvement may be made, as is illustrated in the following examples. In Fig. 12(a) is shown a picture degraded by additive random noise at S/N ratio of 5 dB. In Fig. 12(b) is shown an image obtained by combining the undegraded magnitude with the degraded phase obtained from Fig. 12(a). In Fig. 12(c) is shown an image obtained by combining the degraded magnitude obtained from Fig. 12(a) with the undegraded phase of the original picture. Comparison of Figs. 12(a), (b), and (c) suggest that accurate estimation of the phase as well as the magnitude can be useful in the restoration of images degraded by additive noise.

In this section, we have discussed various image processing problems in which the importance of phase in images and phase-only image reconstruction algorithms have been or have the potential to be exploited. Even though the areas discussed are not exhaustive, they are illustrative of ways in which the importance of phase can be exploited. In our opinion, there is considerable room for further research in understanding these issues.

ACKNOWLEDGMENT

The authors are indebted to a number of colleagues for their assistance and valuable discussions, in particular, Monty Hayes and Gary Kopec.

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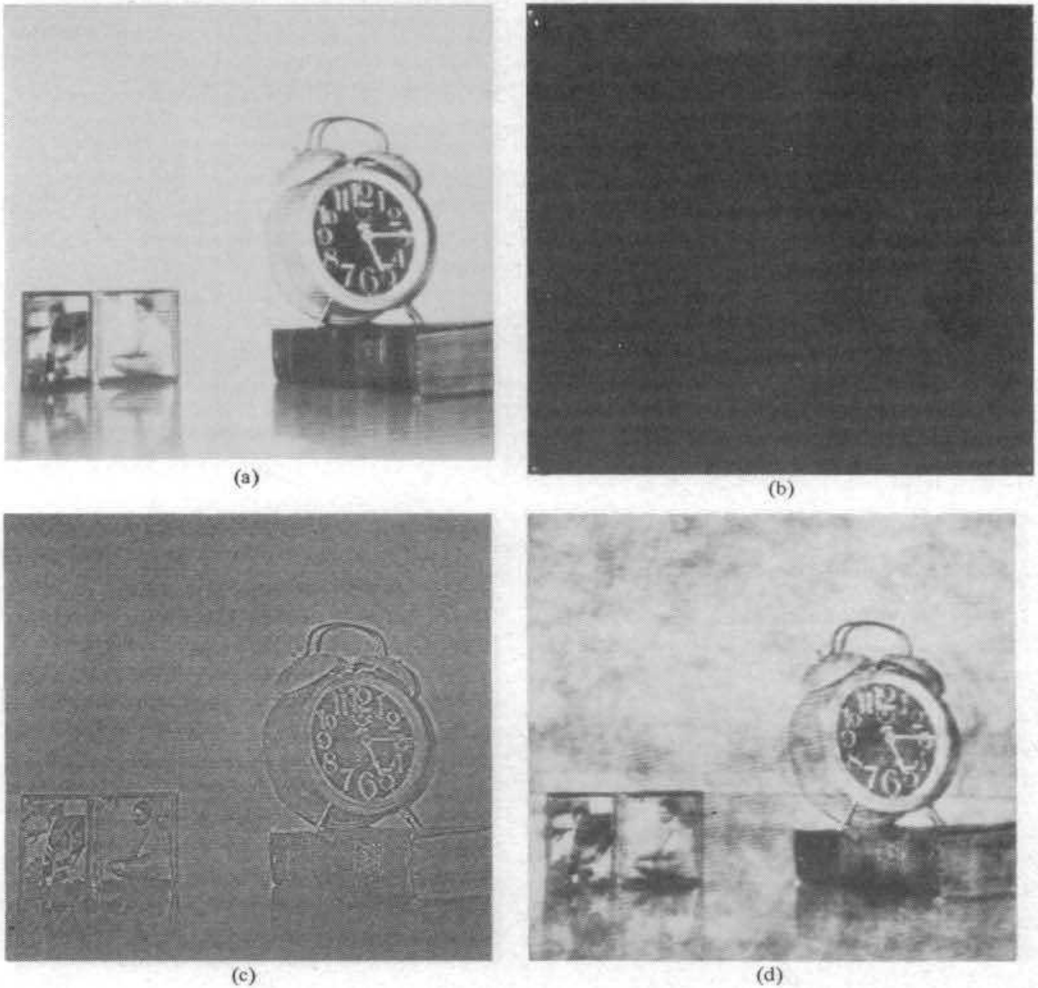


Fig. 2. (a) Original image. (b) Image synthesized from the Fourier transform magnitude of (a) and zero phase. (c) Image synthesized from the Fourier transform phase of (a) and unity magnitude. (d) Image synthesized from the Fourier transform phase of (a) and a magnitude averaged over an ensemble of images.

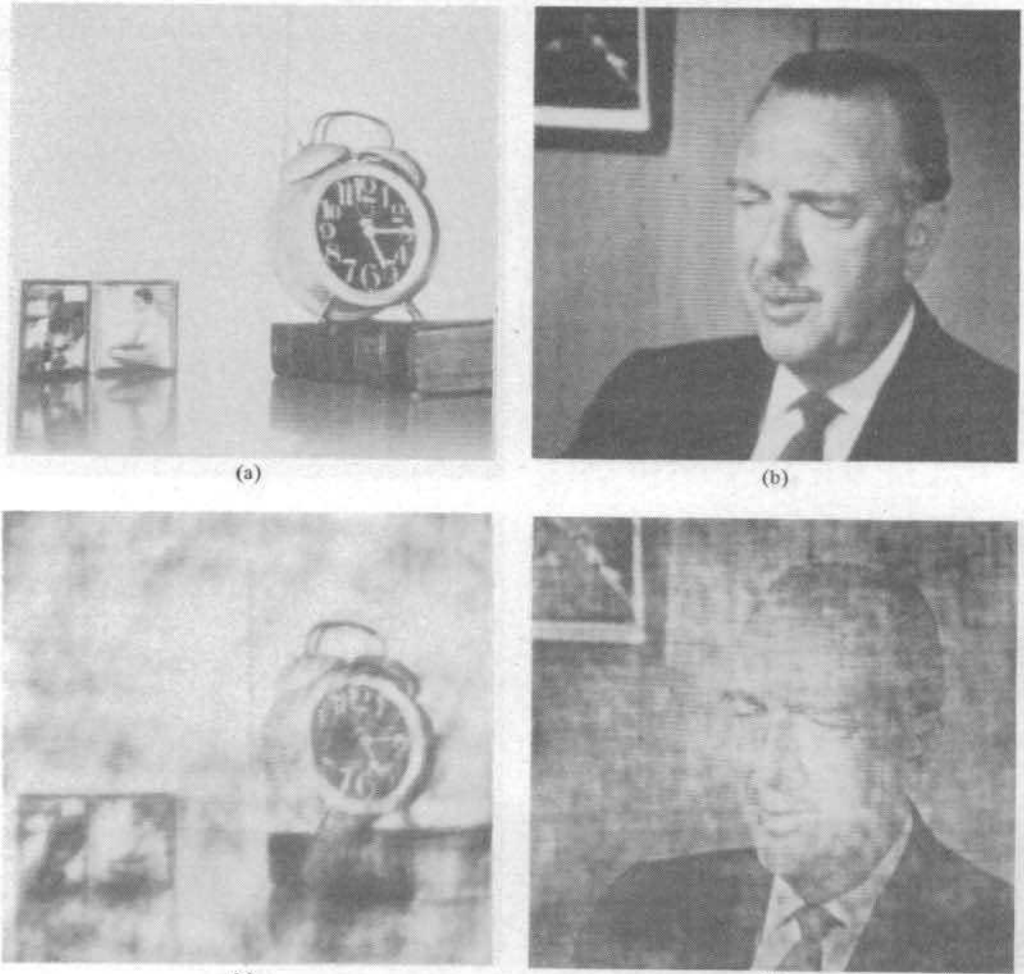
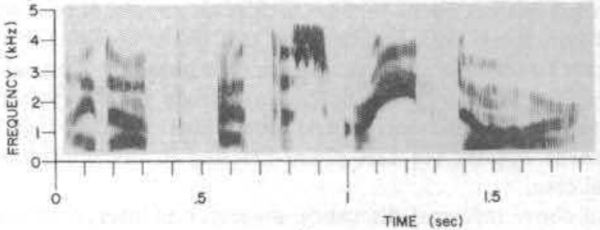
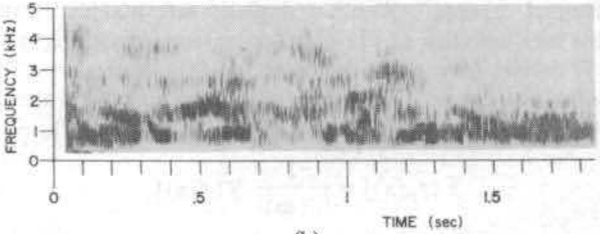


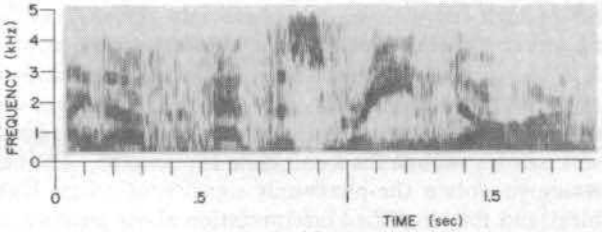
Fig. 3. (a) Original image A. (b) Original image B. (c) Image synthesized from the Fourier transform phase of image A and the magnitude of image B. (d) Image synthesized from the Fourier transform magnitude of image A and the phase of image B.



(a)



(b)



(c)

Fig. 4. (a) Spectrogram of an original sentence "Line up at the screen door." (b) Spectrogram obtained from the Fourier transform magnitude of the entire sentence in (a) and zero phase. (c) Spectrogram obtained from the Fourier transform phase of the entire sentence in (a) and unity magnitude.

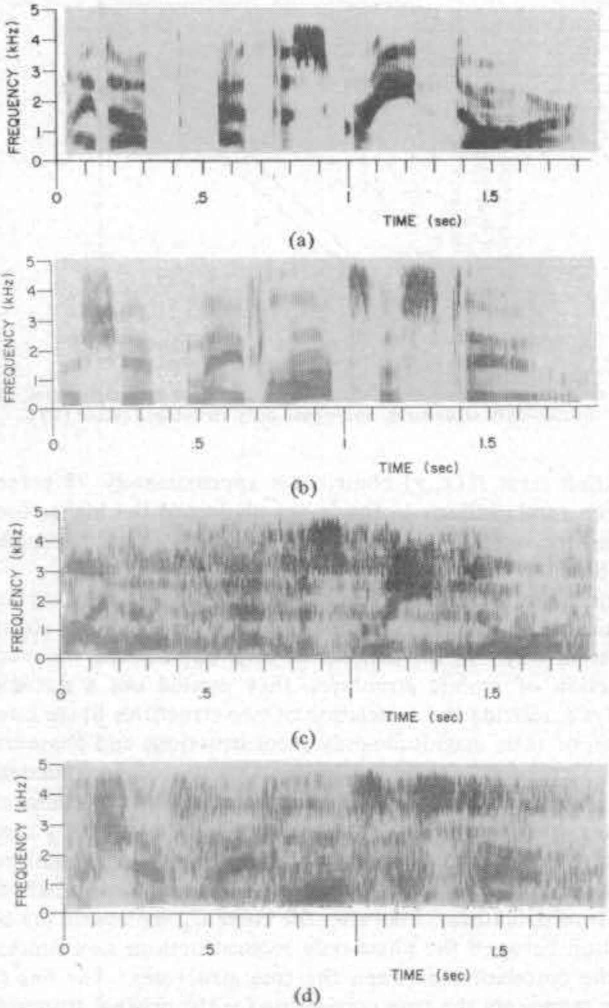


Fig. 5. (a) Spectrogram of original sentence A. (b) Spectrogram of original sentence B. (c) Spectrogram obtained from the Fourier transform phase of sentence A and the magnitude of sentence B. (d) Spectrogram obtained from the Fourier transform phase of sentence B and the magnitude of sentence A.

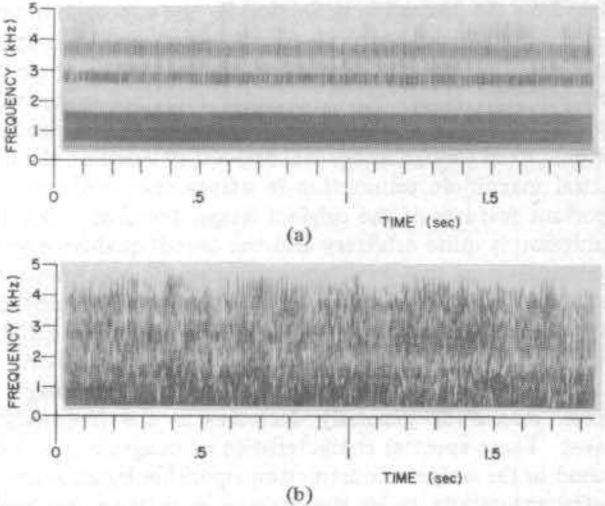
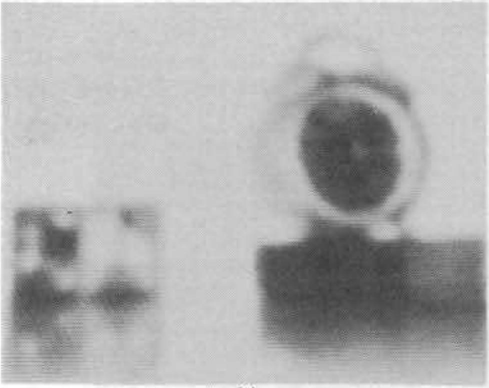
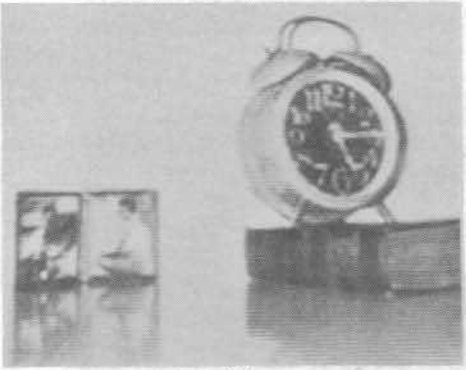


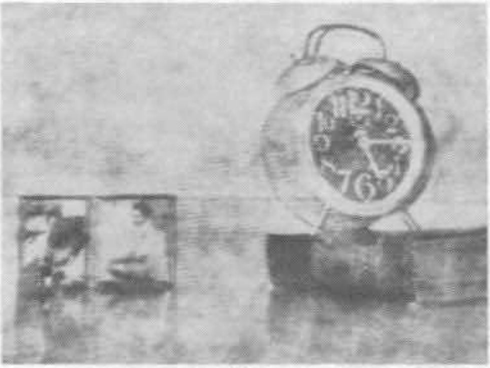
Fig. 7. (a) Spectrogram of a steady-state vowel. (b) Spectrogram obtained from the Fourier transform phase of the steady-state vowel in (a) and unity magnitude.



(a)



(b)



(c)

Fig. 8. (a) The image in Fig. 2(a) blurred by a Gaussian shaped point spread function. (b) Image reconstructed from the Fourier transform phase of the image in (a) and a magnitude estimated from the degraded magnitude. (c) Image reconstructed from the Fourier transform phase of the image in (a) and a magnitude averaged over an ensemble of images.

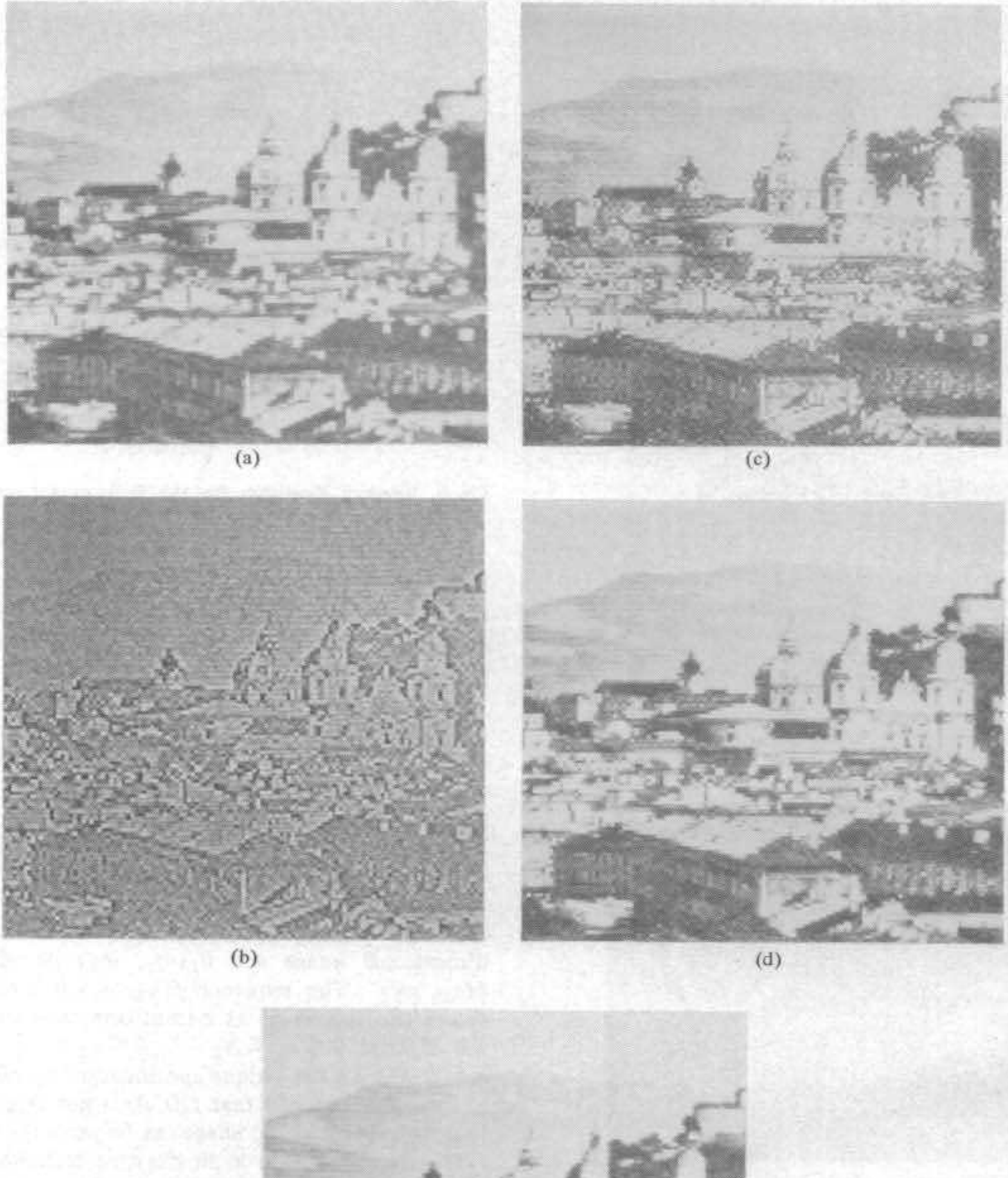
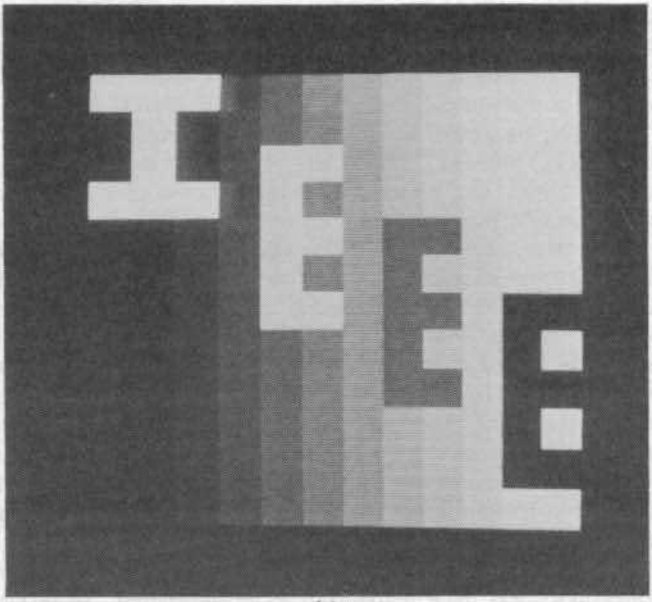
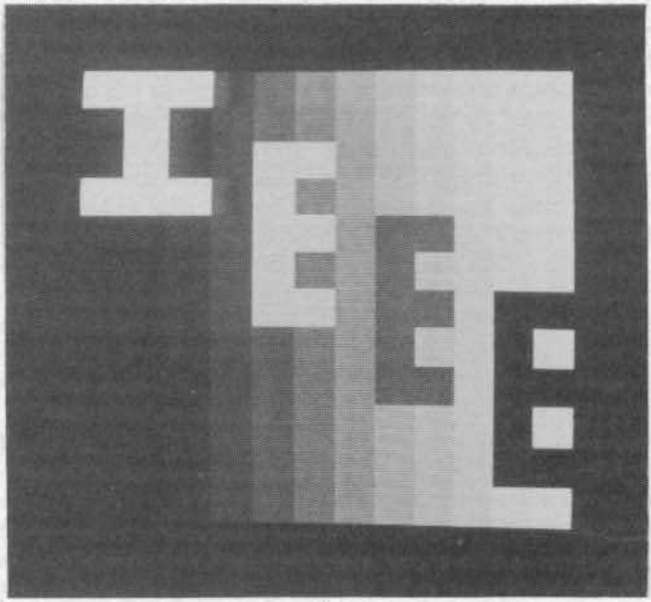


Fig. 10. (a) Original image. (b) Initial image estimate used in the iteration. The estimate was obtained by combining the Fourier transform phase of the image in (a) and unity magnitude. (c) Image reconstructed after 10 iterations. (d) Image reconstructed after 30 iterations. (e) Image reconstructed after 50 iterations.

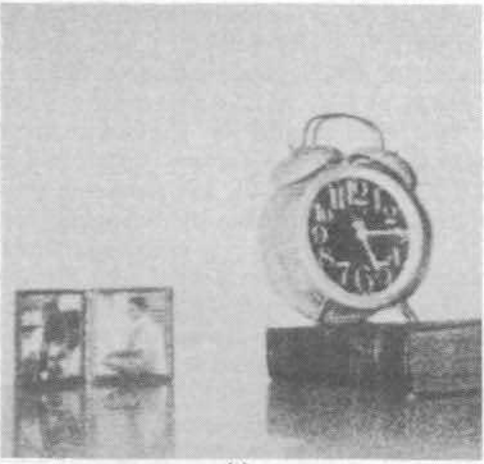


(a)

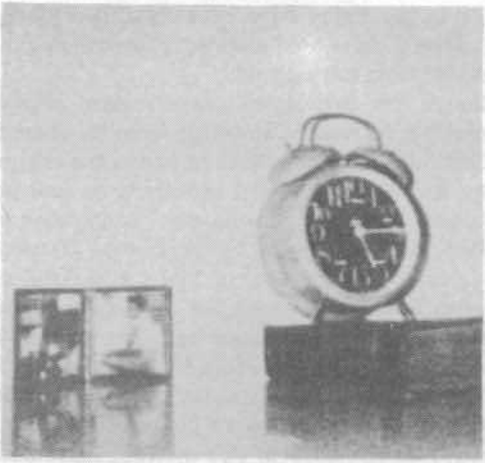


(b)

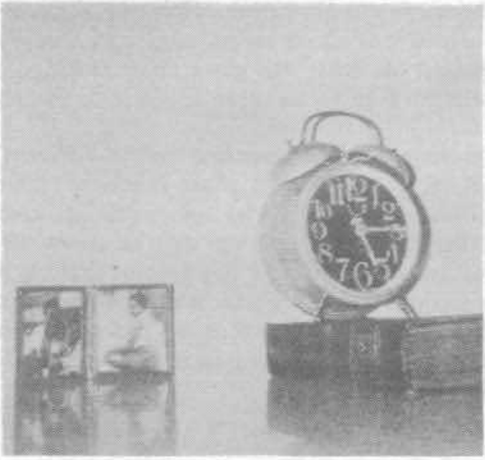
Fig. 11. (a) Original image of 16 × 16 pixels. (b) Reconstructed image from its Fourier transform phase using the closed form solution technique of (13).



(a)



(b)



(c)

Fig. 12. (a) Image degraded by additive white noise at SNR of 5 dB. (b) Image synthesized from the Fourier transform phase of the noisy image in (a) and the magnitude of the original undegraded image. (c) Image synthesized from the Fourier transform magnitude of the noisy image in (a) and the phase of the original undegraded image.