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SUMMARY

The partial group delay between two channels of a multiple time series has an interpretation as the time lag between frequency components of the two channels after adjustments have been made for the influence of the remaining channels. A procedure for estimating partial group delay is proposed, and conditions for consistency and asymptotic normality of the estimating sequence are obtained.

Some key words: Cross spectra; Limit theorem; Multiple time series; Partial coherence; Partial group delay; Partial phase; Time series analysis; Time-lagged relationship.

1. INTRODUCTION

Partial group delay parameters characterize the relationships among the channels of a multiple time series in the following ways. Let X_Λ, Y_Λ and $Z_{1,\Lambda}, \ldots, Z_{p,\Lambda}$ be the spectral components of continuous time processes X, Y and $Z_1, \ldots, Z_p$ for frequencies in a band Λ. Zhang & Foutz (1987) showed that as Λ shrinks to a single frequency λ_0, the relationship between X_Λ and $(Y_\Lambda, Z_{1,\Lambda}, \ldots, Z_{p,\Lambda})$ simplifies to the elementary, linear time-lagged relationship,

$$X_\Lambda(t) = \alpha Y_\Lambda(t-\tau) + \alpha_1 Z_{1,\Lambda}(t-\tau_1) + \ldots + \alpha_p Z_{p,\Lambda}(t-\tau_p) + \varepsilon_\Lambda(t) \quad (-\infty < t < \infty), \qquad (1\cdot1)$$

where the residual process ε_Λ is uncorrelated with Y_Λ, $Z_{1,\Lambda}, \ldots, Z_{p,\Lambda}$. In particular, the time-lag parameter τ in (1·1) is the partial group delay at frequency λ_0 of X behind Y adjusted for $Z_1, \ldots, Z_p$.

In the special case that $p=0$ in (1·1), τ is the unadjusted group delay at λ_0 between X and Y (Deaton & Foutz, 1980a).

Partial group delay parameters are used in the thesis by Deaton and by Deaton & Foutz (1980b) to define causal relationships among frequency components of a multiple time series and to propose a corresponding causal analysis of time series. For example, if the unadjusted group delay parameters τ_{XZ} and τ_{YZ} are both positive in the limiting frequency domain relationships

$$X_\Lambda(t) = \beta_1 Z_\Lambda(t-\tau_{XZ}) + \varepsilon_{XZ,\Lambda}(t), \quad Y_\Lambda(t) = \beta_2 Z_\Lambda(t-\tau_{YZ}) + \varepsilon_{YZ,\Lambda}(t),$$

then Z is a common cause of both X and Y at frequency λ_0. In this case a causal relationship between X and Y, that is not spurious due to the common relationship with Z, is characterized by the partial group delay τ in the relationship

$$X_\Lambda(t) = \alpha Y_\Lambda(t-\tau) + \alpha_1 Z_\Lambda(t-\tau_1) + \varepsilon(t).$$

If $\tau > 0$ then Y causes X at frequency λ_0 after adjusting for the common relationship of both X and Y to Z.

Procedures for estimating partial group delay are required to investigate the time-lagged relationships in (1·1) and to perform causal analyses. Thus, the purpose here is to propose a consistent procedure for estimating partial group delay. The asymptotic distribution of the proposed estimating sequence is also obtained.

Problems of estimating unadjusted group delay are treated by Cleveland & Parzen (1975), Hannan & Thomson (1988, 1981, 1973), Hannan & Robinson (1973), Carter (1981), Chiu (1986), Foutz (1980a), and elsewhere; however, corresponding estimates for partial group delay have yet to be treated in the literature. The procedure of § 2 for estimating partial group delay is closely related to the procedure of Hannan & Thomson (1973) for estimating unadjusted group delay.

2. An estimation procedure

Suppose that $X(t)$, $Y(t)$ and $Z_1(t), \ldots, Z_p(t)$ are zero-mean, weakly stationary processes defined for $-\infty < t < \infty$, and that they have absolutely continuous spectra with continuous spectral densities. The process X, for example, has a spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, dZ_X(\lambda) \quad (-\infty < t < \infty),$$

and the frequency component of X for frequencies in Λ has the spectral representation

$$X_\Lambda(t) = \int_{\Lambda} e^{i\lambda t}\, dZ_X(\lambda) \quad (-\infty < t < \infty).$$

Similar spectral representations exist for $Y, Z_1, \ldots, Z_p$ and for $Y_\Lambda, Z_{1,\Lambda}, \ldots, Z_{p,\Lambda}$.

Partial group delay is defined in terms of partial phase (Koopmans, 1974, p. 156). The Hilbert space L_2 contains random variables having mean zero and finite variances, and the norm of each random variable in L_2 is defined to be its standard deviation. Let H be the closed linear subspace of L_2 that is generated by the random variables $\{Z_1(t), \ldots, Z_p(t); -\infty < t < \infty\}$. It is well known that the processes X and Y can be decomposed into unique processes Π_X and Π_Y, respectively, in H plus processes ε_X and ε_Y that are uncorrelated with the elements of H,

$$X(t) = \Pi_X(t) + \varepsilon_X(t), \quad Y(t) = \Pi_Y(t) + \varepsilon_Y(t).$$

Let the spectral densities and cross spectral density of $\varepsilon_X(t)$ and $\varepsilon_Y(t)$ be $g_{XX}(\lambda)$, $g_{YY}(\lambda)$ and $g_{XY}(\lambda)$. The partial coherence between X and Y adjusted for $Z_1, \ldots, Z_p$ is

$$\sigma(\lambda) = |g_{XY}(\lambda)|/|g_{XX}(\lambda) g_{YY}(\lambda)|^{\frac{1}{2}}.$$

The partial phase is the argument of the complex partial coherence,

$$\phi(\lambda) = \arg[g_{XY}(\lambda)/\{g_{XX}(\lambda) g_{YY}(\lambda)\}^{\frac{1}{2}}].$$

Finally, the partial group delay is the derivative of the partial phase, $\tau(\lambda) = \partial\phi(\lambda)/\partial\lambda$.

Suppose that the N equally spaced observations $X(t)$, $Y(t)$ and $Z_1(t), \ldots, Z_p(t)$ for $t = 1, \ldots, N$ are available for estimating $\tau(\lambda)$ at frequency λ_0. The discrete Fourier transforms of the observations are

$$W_X(\lambda) = (2\pi N)^{-\frac{1}{2}} \sum_{t=1}^{N} X(t)\, e^{i\lambda t}, \quad W_Y(\lambda) = (2\pi N)^{-\frac{1}{2}} \sum_{t=1}^{N} Y(t)\, e^{i\lambda t},$$

$$W_j(\lambda) = (2\pi N)^{-\frac{1}{2}} \sum_{t=1}^{N} Z_j(t)\, e^{i\lambda t} \quad (j = 1, \ldots, p).$$

The discrete Fourier transforms may be used to define the periodograms

$$I_{XX}(\lambda) = W_X(\lambda)\, W_X(\lambda)^c, \quad I_{YY}(\lambda) = W_Y(\lambda)\, W_Y(\lambda)^c, \quad I_{jj}(\lambda) = W_j(\lambda)\, W_j(\lambda)^c,$$

and the cross periodograms

$$I_{X,j}(\lambda) = W_X(\lambda)\, W_j(\lambda)^c, \quad I_{Y,j}(\lambda) = W_Y(\lambda)\, W_j(\lambda)^c, \quad I_{j,k}(\lambda) = W_j(\lambda)\, W_k(\lambda)^c,$$

where, for example, $W_X(\lambda)^c$ is the complex conjugate of $W_X(\lambda)$.

Let B_0 be a band of frequencies centred at λ_0 and containing m of the N fundamental frequencies $\lambda_v = 2\pi v/N$ for integers v $(-\frac{1}{2}(N-1) \leq v \leq \frac{1}{2}N)$. Then, for $M = N/(2m)$,

$$B_0 = \{\lambda : \lambda_0 - \pi/(2M) < \lambda < \lambda_0 + \pi/(2M)\}. \tag{2·1}$$

For $[x]$ the largest integer less than or equal to the real number x, let $L = [\frac{1}{2}p] + 1$, and define the smoothed spectral and cross spectral estimators

$$\hat{f}_{jk}(\lambda) = (2L+1)^{-1} \prod_{l=-L}^{L} I_{jk}(\lambda + l\pi/M), \tag{2·2}$$

with corresponding definitions for $\hat{f}_{XY}$, $\hat{f}_{Xj}$, $\hat{f}_{Yj}$. Corresponding estimates for the cross spectral density, $g_{XY}(\lambda)$, between ε_X and ε_Y, for the spectral density, $g_{XX}(\lambda)$, of ε_X, and for the spectral density $g_{YY}(\lambda)$, of ε_Y take the form for $A = X, Y$ and $B = X, Y$

$$\hat{g}_{AB}(\lambda) = \hat{f}_{AB}(\lambda) - \{\hat{f}_A(\lambda)\}^{\mathrm{T}} \hat{f}_\lambda \hat{f}_B(\lambda)^c, \tag{2·3}$$

where $\hat{f}_A(\lambda)$ is the $p \times 1$ row vector of elements $\hat{f}_{A,j}(\lambda)$ and $\hat{f}_\lambda$ is the $p \times p$ matrix with elements $\hat{f}_{jk}(\lambda)$. Finally, define

$$\hat{p}(\tau) = m^{-1} \textstyle\sum_0 \hat{g}_{XY}(\lambda)\, e^{-i\tau\lambda}, \tag{2·4}$$

where Σ_0 is the sum over the m fundamental frequencies, $\lambda_v = 2\pi v/N$ for integers v, in the band B_0. The proposed estimate of the partial group delay at frequency λ_0 between X and Y adjusted for $Z_1, \ldots, Z_p$ is the value $\hat{\tau}_N$ maximizing $\hat{q}(\tau) = |\hat{p}(\tau)|^2$.

Note that the procedure of Hannan & Thomson (1973) for estimating the unadjusted group delay between X and Y is to take $L = 0$ in (2·2), so that $\hat{f}_{XY}(\lambda) = I_{XY}(\lambda)$, and to define $\hat{p}^*(\tau)$ by replacing $\hat{g}_{XY}(\lambda)$ in (2·4) by $\hat{f}_{XY}(\lambda)$. The Hannan & Thomson (1973) estimate of unadjusted group delay is the τ value maximizing $|\hat{p}^*(\tau)|^2$.

Next, the proposed estimate for partial group delay is shown to be consistent and asymptotically normal.

3. Assumptions and theoretical results

Results on the asymptotic properties of the sequence of partial group delay estimators $\{\hat{\tau}_N\}$ for increasing N require the Conditions A of Hannan & Thomson (1973) that the processes X, Y and $Z_1, \ldots, Z_p$ be ergodic, weakly stationary, and nondeterministic with zero means, positive-definite covariance matrices, and with absolutely continuous spectra having boundedly differentiable spectral densities, and that $\phi(\lambda)$, the partial coherence, be positive. We also require Conditions B of Hannan & Thomson (1973) that ε_X and ε_Y have finite fourth moments and that the fourth cumulant $\kappa_{ijkl}(m, n, p, q)$ for $\varepsilon_i(m)$, $\varepsilon_j(n)$, $\varepsilon_k(p)$ and $\varepsilon_l(q)$ $(i, j, k, l = X, Y)$ be a Fourier transform, namely

$$\kappa_{ijkl}(m, n, p, q) = \int\!\!\int\!\!\int\!\!\int \exp\{i(m\lambda_1 + n\lambda_2 + p\lambda_3 + q\lambda_4)\} f_{ijkl}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\, d\lambda_1\, d\lambda_2\, d\lambda_3\, d\lambda_4,$$

where $|\lambda_j| \leq \pi$ for $j = 1, 2, 3, 4$, where the region of integration contains the points for which $\lambda_1 + \ldots + \lambda_4 = 0$, and where f_{ijkl} is a boundedly differentiable function on this region. For each nonnegative integer u, define

$$c_{XY}(u) = N^{-1} \sum \varepsilon_X(t)\varepsilon_Y(t+u), \quad \gamma_{XY}(u) = E\{\varepsilon_X(t)\varepsilon_Y(t+u)\}.$$

Conditions B also provide for the joint asymptotic normality of the quantities

$$N^{\frac{1}{2}}\{c_{XY}(u) - \gamma_{XY}(u)\}$$

for each fixed u.

Finally, we require Conditions C from Brillinger (1975, Assumption 2.6.1, p. 26): The $p+2$ vector-valued series $X(t)$, $Y(t)$, $Z_1(t), \ldots, Z_p(t)$ is strictly stationary; all moments of its components exist; and all the joint cumulant functions of order k for the vector-valued series are absolutely summable, for $k = 2, 3, \ldots$. The following results are proved in the Appendix.

THEOREM 3·1. *Under Conditions A and C, there exists a sequence of integers* $\{m = N/(2M)\}$, *increasing with N, such that* $\hat{\tau}_N$ *converges in probability to* $\tau(\lambda_0)$.

Define

$$p(\tau) = (2L+1-p)^{-1} \sum_{j=0}^{2L-p} \int_{B_0} g_{XY}(\lambda + j\pi/M)\, e^{-i\lambda\tau}\, d\lambda, \tag{3·1}$$

and let τ_M^* be the value of τ that maximizes $|p(\tau)|^2$ for fixed M.

THEOREM 3·2. *If Conditions A, B and C are met then there exists a sequence of integers* $\{m = N/(2M)\}$, *increasing in N, such that* $m^{3/2}N^{-1}(\hat{\tau}_N - \tau^*)$ *is asymptotically normally distributed with mean zero and variance*

$$\frac{6}{(2L-p+1)\pi^2}\left\{\frac{1-\sigma^2(\lambda_0)}{\sigma^2(\lambda_0)}\right\}, \tag{3·2}$$

where $\sigma(\lambda)$ *is the partial coherence between X and Y adjusted for* $Z_1, \ldots, Z_p$ *and where* $L = [\frac{1}{2}p] + 1$.

4. AN EXAMPLE

The proposed procedure for estimating partial group delay is now illustrated by simulation. For this let $\alpha(t)$, $\beta(t)$, $\eta(t)$ and $\gamma(t)$ be mutually independent zero mean, Gaussian random variables for $t = 0, 1, \ldots$. The variances of $\alpha(t)$, $\beta(t)$, $\eta(t)$ and $\gamma(t)$ are taken to be 0·4, 0·4, 3·0 and 0·06 respectively. Let

$$s(t) = \eta(t) + 0{\cdot}75\eta(t-1),$$

and for $t = 1, 2, \ldots$ construct $\varepsilon_X(t)$ and $\varepsilon_Y(t)$ as

$$\varepsilon_X(t) = s(t) + \alpha(t), \quad \varepsilon_Y(t) = s(t+3) + \beta(t).$$

Finally, construct the time series X, Y and Z by

$$Z(t) = 0{\cdot}5Z(t-1) + \gamma(t), \quad X(t) = 0{\cdot}8Z(t+1) + \varepsilon_X(t),$$
$$Y(t) = 0{\cdot}6Z(t+2) + \varepsilon_Y(t).$$

The autospectra for $\varepsilon_X(t)$ and $\varepsilon_Y(t)$ are

$$g_{XX}(\lambda) = g_{YY}(\lambda) = (2\pi)^{-1}(3|1 + 0{\cdot}75\, e^{i\lambda}|^2 + 0{\cdot}4),$$

and the partial coherence between X and Y adjusted for Z is

$$\sigma(\lambda)=3|1+0\cdot75\ e^{i\lambda}|^2/(3|1+0\cdot75\ e^{i\lambda}|^2+0\cdot4).$$

Since

$$g_{XY}(\lambda)=(2\pi)^{-1}(3|1+0\cdot75\ e^{i\lambda}|^2)\ e^{i3\lambda},$$

the partial phase between X and Y adjusted for Z is $\phi(\lambda)=3\lambda$, and the corresponding partial group delay is $\tau(\lambda)=3$, for $-\pi<\lambda\leq\pi$.

Values for $X(t)$, $Y(t)$ and $Z(t)$ were simulated for $t=1,\ldots,1000$ and used in the proposed estimation procedure with $\lambda_0=\frac{1}{2}\pi$ and with $m=13$ to give the partial group delay estimate $\hat{\tau}_N=1\cdot7$. The simulation was repeated to produce 30 independent values of $\hat{\tau}_N$ at each choice of $m=13$, $m=19$, $m=25$ and $m=31$ in the estimation procedure. The results are reported in Table 1.

Table 1. *Estimates* $\hat{\tau}_N$ *for* $\tau(\frac{1}{2}\pi)=3\cdot0$ *based on simulated samples of size* $N=1000$

No.	$m=13$	$m=19$	$m=25$	$m=31$
1	1·7	3·3	3·0	4·1
2	2·9	3·6	3·5	3·7
3	4·4	4·0	5·4	4·1
4	2·0	4·1	3·3	6·5
5	1·8	2·0	5·9	5·0
6	3·1	3·1	5·2	3·7
7	0·2	4·3	2·5	1·7
8	1·7	2·5	2·6	3·3
9	5·5	2·0	3·0	4·8
10	4·3	5·3	2·3	4·8
11	1·6	1·4	3·9	5·0
12	0·6	5·7	5·6	4·6
13	7·0	4·3	3·1	4·5
14	3·3	2·8	1·1	2·2
15	3·6	0·01	3·0	2·3
16	2·2	5·4	3·5	2·1
17	5·5	2·9	1·0	2·1
18	0·6	2·4	1·8	2·8
19	0·1	2·7	6·4	1·1
20	2·9	0·01	3·9	2·7
21	4·1	2·1	2·5	3·2
22	0·2	3·9	5·8	3·3
23	5·7	2·4	2·9	4·2
24	5·2	7·4	2·5	2·6
25	0·1	2·8	4·3	1·4
26	3·0	3·0	2·5	3·8
27	3·6	0·9	1·9	4·7
28	8·0	3·9	1·0	1·1
29	6·1	1·6	4·0	1·3
30	6·2	4·3	4·5	3·2
Av. $\hat{\tau}_N$	3·24	3·14	3·38	3·33
Lim. mean	3·0	3·0	3·0	3·0
Sample var. of $\hat{\tau}_N$	4·73	2·58	2·09	1·81
Lim. var.	24·6	7·88	3·46	1·82

The asymptotic variance of $m^{3/2}N^{-1}(\hat{\tau}_N - \tau_M^*)$, from Theorem 3·2, is

$$3\{1-\sigma^2(\pi/2)\}/\{\pi^2\sigma^2(\pi/2)\} = 0{\cdot}0541.$$

With $N = 1000$, the corresponding approximate variance, $(N^2/m^3)(0{\cdot}0541)$, of $\hat{\tau}_N$ is $\sigma_{13}^2 = 24{\cdot}6$ for $m = 13$, $\sigma_{19}^2 = 7{\cdot}88$ for $m = 19$, $\sigma_{25}^2 = 3{\cdot}46$ for $m = 25$, and $\sigma_{31}^2 = 1{\cdot}82$ for $m = 31$.

According to Theorem 3·2 the simulated samples for $m = 13$, 19, 25 and 31 should be from approximate normal distributions with means 3·0 and variances σ_{13}^2, σ_{19}^2, σ_{25}^2 and σ_{31}^2 respectively. According to the goodness-of-fit test of Foutz (1980b) the normal fit is not adequate for the sample with $m = 13$ but is marginally adequate for the samples with $m = 19$, 25 and 31. The simulated means of $\hat{\tau}_N$ are all close to the asymptotic means of 3·00. However, when $m = 13$ and the normal approximation is inadequate, the simulated variance of $\hat{\tau}_N$ differs greatly from the asymptotic variance. Further study is required of the relationship between the choice of N and m and the normal approximation.

APPENDIX

Proofs of Theorem 3·1 *and Theorem* 3·2

Assume $\lambda \neq 0 \pmod{\pi}$. The components of the matrix

$$\hat{g}(\lambda) = \begin{bmatrix} \hat{g}_{XX}(\lambda) & \hat{g}_{XY}(\lambda) \\ \hat{g}_{XY}^c(\lambda) & \hat{g}_{YY}(\lambda) \end{bmatrix}$$

are defined in (2·3). Under Conditions C, it follows (Brillinger, 1975, pp. 305–6) that $(2L-p+1)\hat{g}(\lambda)$ is asymptotically distributed as a complex Wishart matrix. Also (Brillinger, 1975, p. 238) a matrix with the same asymptotic distribution is $(2L-p+1)\hat{g}^\varepsilon(\lambda)$, where

$$\hat{g}^\varepsilon(\lambda) = \begin{bmatrix} \hat{g}_{XX}^\varepsilon(\lambda) & \hat{g}_{XY}^\varepsilon(\lambda) \\ \hat{g}_{YX}^\varepsilon(\lambda) & \hat{g}_{YY}^\varepsilon(\lambda) \end{bmatrix}$$

$$= I_{\varepsilon\varepsilon}(\lambda) + I_{\varepsilon\varepsilon}(\lambda + \pi/M) + \ldots + I_{\varepsilon\varepsilon}\{\lambda + (2L-p)\pi/M\},$$

where

$$I_{\varepsilon\varepsilon}(\lambda) = \begin{bmatrix} I_{XX}^\varepsilon(\lambda) & I_{XY}^\varepsilon(\lambda) \\ I_{YX}^\varepsilon(\lambda) & I_{YY}^\varepsilon(\lambda) \end{bmatrix}$$

is the matrix of periodograms $I_{XX}^\varepsilon(\lambda)$, $I_{YY}^\varepsilon(\lambda)$ and cross periodograms $I_{XY}^\varepsilon(\lambda)$, $I_{YX}^\varepsilon(\lambda)$ based directly on the values $\varepsilon_X(t)$, $\varepsilon_Y(t)$ for $t = 1, \ldots, N$.

Let $\lambda_1(N), \ldots, \lambda_m(N)$ be the m fundamental frequencies in the band B_0 in (2·1), and define the following sequences for $N = 1, 2, \ldots$

$$U_{1,N} = \{\hat{g}_{XY}(\lambda_1(N)), \ldots, \hat{g}_{XY}(\lambda_m(N)), 0, 0, \ldots\},$$

$$U_{2,N} = \{\hat{g}_{XY}^\varepsilon(\lambda_1(N)), \ldots, \hat{g}_{XY}^\varepsilon(\lambda_m(N)), 0, 0, \ldots\}.$$

Also, let $W = \{W_1, W_2, \ldots\}$ be an infinite sequence of independent, complex-valued random variables each distributed with the common asymptotic distribution of $\hat{g}_{XY}(\lambda_0)$ and $\hat{g}_{XY}^\varepsilon(\lambda_0)$. It follows from Brillinger (1975, p. 94) that $U_{i,N}$ converges weakly to W, for $i = 1, 2$ (Billingsley, 1968, p. 19).

The proposed estimate for partial group delay is the value $\hat{\tau}_N$ maximizing the function

$$\hat{q}(\tau) = |\hat{p}(\tau)|^2 = |m^{-1} \textstyle\sum_0 \hat{g}_{XY}(\lambda)\, e^{-i\tau\lambda}|^2.$$

To emphasize that this is also a functional on R^∞, write

$$\hat{q}(\tau) = h_N(\tau, U_{1,N}).$$

An estimate $\hat{\tau}_N^\varepsilon$ for partial group delay based directly on the values $\varepsilon_X(t)$ and $\varepsilon_Y(t)$ for $t=1,\ldots,N$ is the value of τ that maximizes $h_N(\tau, U_{2,N})$.

Since $\hat{\tau}_N$ satisfies $\partial h_N(\tau, U_{1,N})/\partial\tau=0$, a form of the Implicit Function theorem (Brillinger, 1975, pp. 75-6) shows that $\hat{\tau}_N$ has the representation $\hat{\tau}_N=\Xi_N(U_{1,N})$ for some continuous function Ξ_N of the first N components of $U_{1,N}$. For the same reasons $\hat{\tau}_N^\varepsilon$ has the representation $\hat{\tau}_N^\varepsilon=\Xi_N(U_{2,N})$.

Proofs of Hannan & Thomson (1973) show that, under Conditions A, $\tau_N^\varepsilon=\Xi(U_{2,N})$ converges weakly to the degenerate distribution at $\tau(\lambda_0)$. Since $U_{2,N}$ and $U_{1,N}$ both converge weakly to W, it follows (Topsoe, 1967, Th. 2) that $\hat{\tau}_N=\Xi(U_{1,N})$ also converges weakly to the degenerate distribution at $\tau(\lambda_0)$, and this proves Theorem 3·1.

Write

$$h_N(U_{i,N})=m^{3/2}N^{-1}\{\Xi_N(U_{i,N})-\tau_M^*\}\quad(i=1,2).$$

Since under Conditions A and B $m^{3/2}N^{-1}(\hat{\tau}^\varepsilon-\tau_M^*)=h_N(U_{2,N})$ converges weakly to a zero-mean Gaussian random variable Q', and since $U_{2,N}$ and $U_{1,N}$ both converge weakly to W, it follows (Topsoe, 1967, Th. 2) that $m^{3/2}N^{-1}(\hat{\tau}_N-\tau_M^*)=h_N(U_{1,N})$ also converges weakly to Q'. This proves the asymptotic normality asserted in Theorem 3·2.

The proof that the variance of Q' is (3·2) closely follows the proof of Theorem 2 of Hannan & Thomson (1973).

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