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SEEING THE FISHER Z-TRANSFORMATION

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Abstract

Since 1915, statisticians have been applying Fisher's Z-transformation to Pearson product-moment correlation coefficients. We offer new geometric interpretations of this transformation.

Key words: correlation coefficient, Fisher, geometry, hyperbolic, transformation

1. Introduction

Noting some limitations of Pearson's product-moment correlation coefficient (r), Fisher (1915) suggested a transformation

$$Z_r = \text{arctanh}(r)$$

that has advantages over r. Relative to the correlation coefficient, Z_r has a simpler distribution; its variance is more nearly independent of the corresponding population parameter (Z_ρ); and it converges more quickly to normality (Johnson, Kotz, and Balakhrishnan, 1995). Fisher's Z transformation is featured in statistics texts (e.g., Casella and Berger, 2002) and is used by meta-analysts (Lipsey and Wilson, 2001).

Much has been learned about Z_r since 1915. We now know the exact distribution of Z_r for data from a bivariate normal distribution (Fisher, 1921), the exact distribution of Z_r for data from a bivariate Type A Edgeworth distribution (Gayen, 1951), and the asymptotic distribution of Z_r for virtually any data (Hawkins, 1989). We know that Z_r can be derived as a variance-stabilizing transformation or a normalizing transformation (Winterbottom, 1979). We have Taylor series expressions for the moments of Z_r and several related statistics (Hotelling, 1953).

Although scholars have been thorough in describing the analytic properties of Z_r, they have had little to say about the geometry of this transformation. True, there is a geometric flavor to certain discussions of Z_r -transformed correlation matrices (Brien, Venables, James, and Mayo, 1984). Still, the dearth of geometric knowledge about Z_r is striking, when geometric treatments of r abound (Rodgers and Nicewander, 1988).

In the current article, we offer the first geometric interpretations of Z_r to date. In Section 2, we develop some Euclidean area representations. These depict r and Z_r as areas — both in the two-dimensional scatterplot, and in an N-dimensional vector space.

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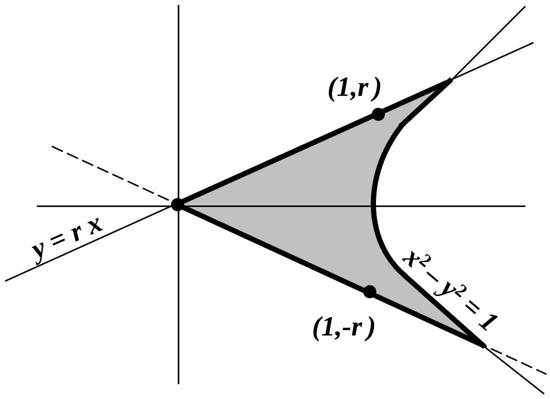

FIGURE 1.
Fisher's Z_r as area in scatterplot

Line segments bound the areas that represent r. Line segments and hyperbolas bound the areas that represent Z_r . Our area depictions of Z_r are easy to visualize; however, the corresponding depictions of r are non-standard. In Section 3, we introduce some concepts from hyperbolic geometry which are surprisingly useful in understanding Z_r. These allow us to develop analogues to the usual geometric representations of r. There, we interpret Z_r as a slope of the least-squares regression line in a two-dimensional scatterplot and as the length of the projection of one N-dimensional vector onto another. We also identify an error criterion that is compatible with Z_r. In Section 4, we make a few final observations.

2. Z_r as Euclidean area

2.1. Areas in two dimensions

Correlational statistics can be represented as areas in Euclidean space. Suppose that we have data on two variables (X and Y) which we have standardized in the usual manner via $x_i = \frac{(X_i - \overline{X})}{s_X}$ and $y_i = \frac{(Y_i - \overline{Y})}{s_Y}$. Then the least-squares regression line for predicting y from x is, of course, $y = rx$. Let us depict this regression line in a two-dimensional xy scatterplot, along with its reflection in the x-axis — the line $y = -rx$. Let us also insert into our xy scatterplot the *unit hyperbola* $\mathbf{H}^1 = \{(x, y) \mid x^2 - y^2 = 1,\, x > 0\}$. The quantity Z_r can be regarded as the area enclosed by this hyperbola, the regression line, and its reflection. See Figure 1.

To justify this representation, let us begin by changing to polar coordinates $(x, y) = (u \cos\phi, u \sin\phi)$. Then the equation of the hyperbola becomes $x^2 - y^2 = u^2\left(\cos^2\phi - \sin^2\phi\right) = u^2 \cos(2\phi) = 1$, so that the area indicated in Figure 1 can be expressed as the integral

$$\frac{1}{2}\int_{-\arctan(r)}^{\arctan(r)} u(\phi)^2\, d\phi = \frac{1}{2}\int_{-\arctan(r)}^{\arctan(r)} \frac{1}{\cos(2\phi)} d\phi$$

$$= \frac{1}{2}\ln\left(\frac{r+1}{r-1}\right) = \operatorname{arctanh}(r) = Z_r.$$

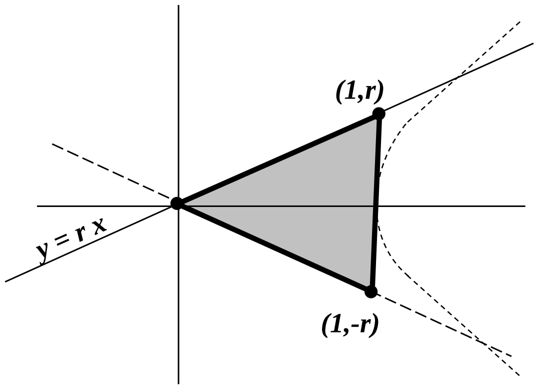

FIGURE 2.
Pearson's r as area in scatterplot

It is possible to represent the Pearson product-moment correlation coefficient in a similar picture. We do so by replacing the hyperbola from Figure 1 with a vertical line drawn at $x = 1$. Now r can be depicted as the signed area of a triangle – the triangle formed by the least-squares regression line $y = rx$, its reflection $y = -rx$, and the line $x = 1$. See Figure 2.

These area representations illustrate certain features of Fisher's Z transformation. When the least-squares regression line is horizontal, both r and Z_r are 0. The two relevant "areas" of the scatterplot are degenerate because the regression line $y = rx$ coincides with its reflection $y = -rx$. When the regression line has non-zero slope, the area representing r is contained in the area representing Z_r ; thus, $|r| < |Z_r|$. Note that these areas are similar in size when the regression line is nearly horizontal and diverge as the line becomes steeper. In the extreme case, $r = \pm 1$, the least-squares regression line and its reflection are the asymptotes $y = \pm x$ of the hyperbola $x^2 - y^2 = 1$, and Z_r is unbounded.

2.2. Areas in N dimensions

Data that can be represented as N points in 2-space can also be represented as two vectors in N-space. In N-dimensional space, the Pearson product-moment correlation coefficient between two variables can be regarded as the cosine of the angle between the corresponding vectors, as Fisher (1915) first noted.

We exploit this construction in Figure 3. We now represent the data by the two normalized vectors $\mathbf{x} = \frac{\left(X_1 - \overline{X}, \ldots, X_N - \overline{X}\right)}{\sqrt{\sum \left(X_i - \overline{X}\right)^2}}$ and $\mathbf{y} = \frac{\left(Y_1 - \overline{Y}, \ldots, Y_N - \overline{Y}\right)}{\sqrt{\sum \left(Y_i - \overline{Y}\right)^2}}$ in $\mathbf{R}^N$ and consider the perpendicular projection of $\mathbf{y}$ onto $\mathbf{x}$, which we symbolize by $P_{\mathbf{x}}(\mathbf{y})$. Since each vector has been normalized to length 1, r is the length of $P_{\mathbf{x}}(\mathbf{y})$ — as we will be discussing below. In the meantime, let us note that Z_r can be represented by an area in the two-dimensional span of $\mathbf{x}$ and $\mathbf{y}$.

To construct this area, we begin by placing two axes onto this subspace – one in the direction of $\mathbf{x}$ (which we call u) and a second orthogonal to it (which we call v). We position the axes so that $\mathbf{x}$ and $\mathbf{y}$ originate at the point $(1, 0)$ in the (u, v) coordinate system. We construct the line $u = 1 + r$, which is coincident with the perpendicular from

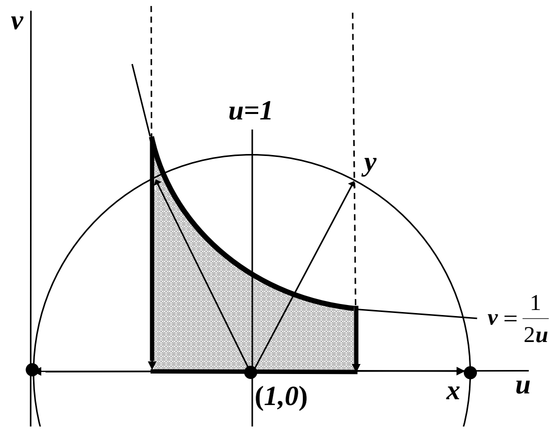

FIGURE 3.
Fisher's Z_r as area in N-space

$\mathbf{y}$ to $\mathbf{x}$. Next we reflect both $\mathbf{x}$ and $\mathbf{y}$ about the line $u = 1$, drop a perpendicular from the reflection of $\mathbf{y}$ onto the reflection of $\mathbf{x}$, and extend this perpendicular to form the line $u = 1 - r$. We finish the construction by placing into this subspace the hyperbola $v = \frac{1}{2u}$, which is the unit hyperbola rotated by $\frac{\pi}{4}$. See Figure 3. Then the signed area between this hyperbola and the u-axis bounded by the two lines $u = 1 - r$ and $u = 1 + r$ is

$$\int_{1-r}^{1+r} \frac{1}{2u} du = \frac{1}{2} \ln(1+r) - \frac{1}{2} \ln(1-r) \tag{1}$$

$$= \frac{1}{2} \ln\left(\frac{1+r}{1-r}\right) \tag{2}$$

$$= \operatorname{arctanh}(r) = Z_r. \tag{3}$$

Having depicted Z_r as a two-dimensional area in N-space, we can get a picture of r by making a single modification to Figure 3: we replace the hyperbola $v = \frac{1}{2u}$ with the line $v = -\frac{1}{2}u + 1$ (the tangent line to the hyperbola at $u = 1$). See Figure 4. Then this line bounds a trapezoid whose three other sides are the u-axis, the line $u = 1 - r$, and the line $u = 1 + r$. This trapezoid has an area of r, as should be apparent because this area would be unchanged if we rotated the line $v = -\frac{1}{2}u + 1$ counterclockwise about the point $\left(1, \frac{1}{2}\right)$ until it became the horizontal line $v = \frac{1}{2}$. The area of the resulting rectangle is clearly r. We prefer the trapezoid because it is contained within the area representing Z_r. Also, observe that the trapezoidal area r is precisely the midpoint approximation with one subdivision to the integral of equation (3) above representing the area Z_r.

Let us note how Figures 3 and 4 illustrate Fisher's Z transformation. When $r = 0$, $\mathbf{y}$ is orthogonal to $\mathbf{x}$, and $P_{\mathbf{x}}(\mathbf{y})$ has 0 length. Thus, the two vertical sides of the trapezoid in Figure 4 are coincident, and r is represented by a degenerate figure that has no area. The Z_r area is also degenerate, it too having zero width. As r diverges from zero, the two bounding lines diverge, and it becomes relevant to note that the hyperbola $v = \frac{1}{2u}$ lies above the line $v = -\frac{1}{2}u + 1$, so that $|r| < |Z_r|$ for all nonzero r. In the extreme case (when $r = \pm 1$), $\mathbf{y}$ is superimposed over $\mathbf{x}$, the line $u = 1 \mp r$ never intersects the hyperbola of Figure 3, and Z_r is unbounded.

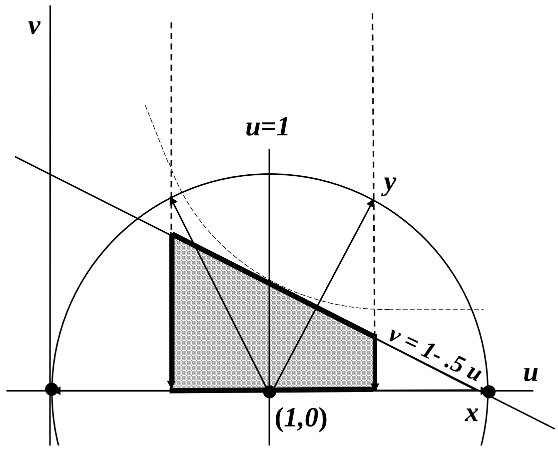

FIGURE 4.
Pearson's r as area in N-space

2.3. Z_r- and r-inspired geometry

Having depicted Z_r as a Euclidean area in two-dimensional and N-dimensional space, let us comment on our representations. These depictions of Z_r are easy to understand, because of the familiarity of Euclidean geometry. Moreover, each of our Euclidean area depictions of Z_r is directly comparable to a depiction of r in the same space; hence, the relationship between the two statistics can be readily seen. These representations have a drawback, however. The correlation coefficient is rarely represented as an area, and our pictures of r may seem a bit contrived. In fact, we began the work described in Sections 2.1 and 2.2 by developing geometric representations of Z_r and (having constructed Z_r) then sought parallel depictions of r. We now reverse the logic of these constructions. We begin Section 3 with two standard geometric representations of r – one in two dimensions and a second in N dimensions. We then create analogous representations of Z_r. To do so, we must leave Euclidean space and introduce some concepts from hyperbolic geometry.

3. Z_r in hyperbolic space

In the last century (or so), mathematicians have developed alternatives to the geometry described by Euclid thousands of years ago. One of these — hyperbolic geometry — is uniquely well suited for representing Fisher's Z_r. Here we depict Z_r with two models of hyperbolic geometry. Each model will be defined on a certain subset of $\mathbf{R}^n$. On each space, we define a distance function from which properties of the model can be deduced. We have a special interest in distances, angles, and geodesics (that is, distance-minimizing curves).

3.1. Euclidean slope and hyperbolic slope

For the most common geometric interpretation of r, the two variables of interest (X and Y) are standardized to x and y as in Section 2.1 above, and depicted as N points in a two-dimensional scatterplot. Figure 2 represented r as an area in this plot; but r is usually regarded as a slope — the slope of the least-squares regression line of y on x. For a data point that has a horizontal distance from the origin of one unit (that is, one standard deviation), we predict that its vertical distance from the origin will be r

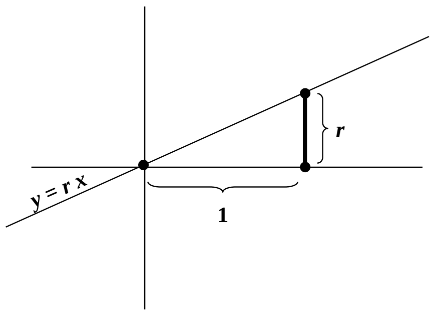

FIGURE 5.
Pearson's r as Euclidean slope

units (that is, r standard deviations). All distances are, of course, defined in a Euclidean metric. Thus, r is the Euclidean "rise" over the Euclidean "run" of the standardized least-squares regression line. See Figure 5.

In this Section, we develop an analogous interpretation for Z_r by showing that Fisher's Z transform can be regarded as the hyperbolic slope of the standardized least-squares regression line. In particular, Z_r can be seen as the hyperbolic "rise" of the regression line over its Euclidean "run."

For this interpretation, we use a one-dimensional model of hyperbolic space – the unit hyperbola $\mathbf{H}^1 = \{(x, y) \mid x^2 - y^2 = 1,\ x > 0\}$. Recall that the $\mathbf{H}^1$ can be parametrized as $(x, y) = (\cosh(t), \sinh(t)) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$ for $t \in \mathbf{R}$, that $\tanh(t) = \frac{\cosh(t)}{\sinh(t)}$, and that the associated inverse functions are denoted arctanh, arccosh, and arcsinh.

Our construction also requires a distance function on $\mathbf{H}^1$. For convenience, we parametrize the hyperbola by

$$\begin{aligned}(x, y) &= \left(\frac{1}{\sqrt{1-t^2}}, \frac{t}{\sqrt{1-t^2}}\right) \\ &= (\cosh(\operatorname{arctanh}(t)), \sinh(\operatorname{arctanh}(t)))\end{aligned}$$

for $-1 < t < 1$. Let two points on the unit hyperbola be designated $P = \left(\frac{1}{\sqrt{1-a^2}}, \frac{a}{\sqrt{1-a^2}}\right)$ and $Q = \left(\frac{1}{\sqrt{1-b^2}}, \frac{b}{\sqrt{1-b^2}}\right)$. The hyperbolic arc on $\mathbf{H}^1$ between P and Q can be parametrized by the curve $\alpha(t) = \left(\frac{1}{\sqrt{1-t^2}}, \frac{t}{\sqrt{1-t^2}}\right)$ for $a \le t \le b$. Then we use an inner product (the *hyperbolic metric*) that is defined as $\langle v, w\rangle = -v_1w_1 + v_2w_2$ to compute the hyperbolic length of this arc, the distance $d(P, Q)$ from P to Q, as follows. We have $\alpha'(t) = \left(\frac{t}{(1-t^2)^{3/2}}, \frac{1}{(1-t^2)^{3/2}}\right)$, and

$$\begin{aligned}d(P, Q) &= \text{hyperbolic arclength from } P \text{ to } Q &(4)\\ &= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t)\rangle}\, dt &(5)\\ &= \int_a^b \sqrt{-\frac{t^2}{(1-t^2)^3} + \frac{1}{(1-t^2)^3}}\, dt &(6)\end{aligned}$$

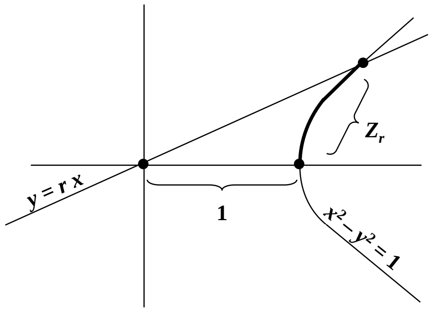

FIGURE 6.
Fisher's Z_r as hyperbolic slope

$$= \int_a^b \frac{1}{1-t^2}\,dt = \operatorname{arctanh}(b) - \operatorname{arctanh}(a) \tag{7}$$

Note that the point $P = (1, 0) = (\cosh(0), \sinh(0))$ is in $\mathbf{H}^1$ and can be viewed as the origin of $\mathbf{H}^1$, in the following sense. By (7), the hyperbolic distance from any point $\left(\frac{1}{\sqrt{1-t^2}}, \frac{t}{\sqrt{1-t^2}}\right)$ in $\mathbf{H}^1$ to the origin $(1, 0)$ is $\operatorname{arctanh}(t) - \operatorname{arctanh}(0) = \operatorname{arctanh}(t)$. For details about this hyperboloid model of hyperbolic space, see Cannon, Floyd, Kenyon, and Parry (1997) and Bridson and Haefliger (1999). It is more commonly used in higher dimensions.

We are now prepared to interpret Z_r as a hyperbolic slope. To do so, we insert $\mathbf{H}^1$ into the xy scatterplot. Note that at the point where the regression line $y = rx$ intersects $\mathbf{H}^1$, we have $x^2 - (rx)^2 = 1$ and $x > 0$, or $x = \frac{1}{\sqrt{1-r^2}}$, $y = \frac{r}{\sqrt{1-r^2}}$. By the arclength expression in equation (7) above, it is evident that the hyperbolic rise of this intersection point (from the horizontal axis) is $\operatorname{arctanh}(r) = Z_r$.

Thus, Z_r can be regarded as the hyperbolic "rise" of the regression line corresponding to a Euclidean "run" of one unit; or as the *hyperbolic slope* of the line $y = rx$. See Figure 6.

A standardized least-squares regression line has a single well-defined hyperbolic slope. A line's hyperbolic slope does not depend on the position along the line at which we begin to measure its slope, nor on the "run" from which the slope is computed. To see that the hyperbolic slope is well-defined, consider a line $y - y_0 = m(x - x_0)$ whose Euclidean slope m is between 1 and -1 and which contains the point (x_0, y_0). Choose any nonzero Δx, which corresponds to the Euclidean run. Next, consider the hyperbola that has a vertical tangent at $(x_0 + \Delta x, y_0)$ and whose asymptotes intersect at (x_0, y_0) and have slopes ± 1 — namely, the hyperbola $(x - x_0)^2 - (y - y_0)^2 = (\Delta x)^2$. If we intersect this hyperbola with the line $y - y_0 = m(x - x_0)$, the result is the point $\left(x_0 + \frac{\Delta x}{\sqrt{1-m^2}}, y_0 + \frac{m\Delta x}{\sqrt{1-m^2}}\right)$. Using the hyperbolic metric $\langle v, w\rangle = -v_1 w_1 + v_2 w_2$ to compute the hyperbolic arclength from this point to $(x_0 + \Delta x, y_0)$— that is, the hyperbolic "rise"— calculations similar to the above yield (assume $\Delta x > 0$ for simplicity)

$$\begin{aligned} \text{hyperbolic}\quad \text{rise} &= \Delta x \int_0^m \sqrt{-\frac{t^2 (\Delta x)^2}{(1-t^2)^3} + \frac{(\Delta x)^2}{(1-t^2)^3}}\,dt \\ &= \Delta x \operatorname{arctanh}(m), \end{aligned}$$

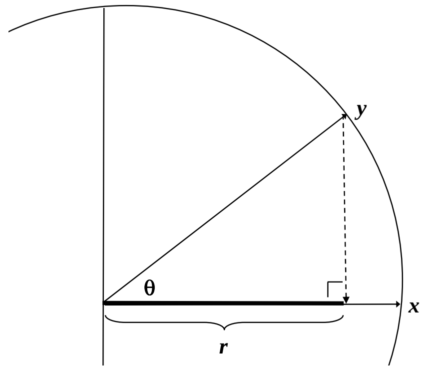

FIGURE 7.
Pearson's r as Euclidean length

so that the hyperbolic slope still calculates as

$$\begin{aligned}\text{hyperbolic} \quad \text{slope} :&= \frac{\text{hyperbolic} \quad \text{rise}}{\text{Euclidean} \quad \text{run}} = \frac{\Delta x \operatorname{arctanh}(m)}{\Delta x} \\ &= \operatorname{arctanh}(m)\,.\end{aligned}$$

This is the inverse hyperbolic tangent of the ordinary slope, which is independent of the choice of the point on the line and independent of Δx.

This representation illustrates features of Z_r that should now be familiar. When the regression line of y on x is horizontal, it intersects the unit hyperbola at the point $(1, 0)$, and the line has no hyperbolic distance from the x-axis. Thus a hyperbolic slope of 0 corresponds to a Euclidean slope of 0. As the Euclidean slope increases, so does the hyperbolic slope. In the extreme case, $r = \pm 1$, the standardized regression line is asymptotic to the unit hyperbola, and the hyperbolic "rise" of the line is undefined.

3.2. Euclidean projection and hyperbolic projection

Fisher (1915) created the usual N-dimensional depiction of the correlation coefficient. Let $\mathbf{x}$ and $\mathbf{y}$ be the two N-dimensional data vectors of Section 2.2, and note that these vectors lie on the unit sphere. Figure 4 depicted r as an area in the subspace spanned by $\mathbf{x}$ and $\mathbf{y}$, but it is simpler to give r a length interpretation. In particular, the correlation between X and Y is the cosine of the angle θ between $\mathbf{y}$ and $\mathbf{x}$. When $\mathbf{y}$ is normalized to have a length of 1, r is the signed length of the perpendicular projection of $\mathbf{y}$ onto $\mathbf{x}$. In the notation of Section 2.2 above, $|r| = |P_{\mathbf{x}}(\mathbf{y})|$. On the other hand, the projection of $\mathbf{y}$ onto the line perpendicular to $\mathbf{x}$ yields the vector connecting $P_{\mathbf{x}}(\mathbf{y})$ and $\mathbf{y}$; the length of this is $|\mathbf{y} - P_{\mathbf{x}}(\mathbf{y})| = \sin\theta = \sqrt{1 - r^2}$. This quantity measures the lack of fit of the least-squares line. All lengths are, of course, Euclidean. See Figure 7.

For an analogous picture of Z_r, we must enter hyperbolic N-space. Thus, consider a non-Euclidean metric on the open unit ball in $\mathbf{R}^N$, for which the curves defined by the vectors $\mathbf{x}$ and $\mathbf{y}$ are infinite geodesic rays separated by an angle θ in hyperbolic space. This ball model of hyperbolic N-dimensional space of constant sectional curvature -1 is the subset $B^N = \{(x_1, ..., x_N) \mid |(x_1, ..., x_N)| < 1\} \subset \mathbf{R}^N$, endowed with the metric $\langle v, w\rangle_{(x_1,\ldots,x_N)} = \left(\frac{4}{\left(1 - x_1^2 - \ldots - x_N^2\right)^2}\right) v \cdot w$ for vectors v and w originating at $(x_1, ..., x_N)$, where $v \cdot w$ is the usual dot product. It turns out that the geodesics (length-minimizing curves) in this metric are either lines through the origin or circles perpendicular to the unit sphere $S^N = \{(x_1, ..., x_N) \mid |(x_1, ..., x_N)| = 1\}$, and all of these geodesics are infinitely long. Two such geodesics are called *asymptotic* if the corresponding circles or lines meet at a point on the unit sphere. The directed line segments connecting the origin to the normalized data points $\mathbf{x}$ and $\mathbf{y}$ are infinite geodesic segments (call them α and β, respectively), and the plane through the origin containing the segments is isometric to two-dimensional hyperbolic space. Since this metric is conformal to the Euclidean metric, the hyperbolic angles between curves in this hyperbolic model are exactly the Euclidean angles between the curves. See, for example, Cannon, Floyd, Kenyon, and Parry (1997) for facts about this and other models of hyperbolic space.

In Euclidean N-space, Pearson's r is the length of the perpendicular projection of one normalized vector onto another. In hyperbolic N-space, Fisher's Z_r has a parallel interpretation, as we now explain. Consider two unit vectors v and w starting at a point in the N-dimensional hyperbolic space of constant sectional curvature -1, and let α and β be the unit-speed geodesics with initial velocities v and w, respectively. Suppose the angle between the two vectors is θ. Next, form the asymptotic geodesic right triangle defined as follows. The first infinite side (the hypotenuse) consists of the points $\alpha(t)$ for $0 \leq t \leq \infty$. The finite side consists of the points $\beta(t)$ for t between 0 and T for a fixed $T \neq 0$, to be determined later. The infinite leg of this triangle is a geodesic that is asymptotic to the hypotenuse β, that contains the point $\beta(T)$, and that is perpendicular to the geodesic β at $\beta(T)$. There is a unique T that allows these conditions to be satisfied. We wish to find T in terms of θ; observe that $|T|$ is the length of the finite side. The quantity T is called *the hyperbolic projection of* α *onto* β, which may be positive or negative.

There are many ways to calculate this quantity T; we choose a coordinate-free method. Observe that any (possibly asymptotic) geodesic triangle with leg $|T|$, opposite angle B, and other angles A and C satisfy the angular hyperbolic Law of Cosines equation $\cosh(T) = \frac{\cos B + \cos A \cos C}{\sin A \sin C}$; see, for example, Anderson (1999, section 5.7). Letting $C = \frac{\pi}{2}$, $B = 0$, and $A = \theta$ as in our case, we obtain the equation $\cosh(T) = \csc(\theta)$. We then obtain the equation

$$\sqrt{\cosh^2(T) - 1} = \sqrt{\csc^2(\theta) - 1}, \quad \text{or}$$
$$\sinh T = \cot\theta,$$

noting that T is negative if $\theta > \frac{\pi}{2}$. Dividing this equation by the original equation of $\cosh(T)$, we get

$$\tanh(T) = \cos\theta, \quad \text{or}$$
$$T = \operatorname{arctanh}(\cos\theta)$$
$$= \operatorname{arctanh}(r) = Z_r,$$

if the geodesic rays α and β correspond to the normalized Euclidean data vectors $\mathbf{x}$ and $\mathbf{y}$, respectively. Therefore, the hyperbolic projection of the end $\alpha(\infty)$ onto the geodesic β is the Fisher Z transform corresponding to the correlation coefficient $r = \cos\theta$. Note that the Euclidean position of the point $\beta(a)$ relative to the origin is $\frac{1-\sin\theta}{\cos\theta}$ units in the direction of the velocity vector of β, and the infinite geodesic is an arc of the Euclidean circle tangent to α and perpendicular to β. See Figure 8.

This figure, like the earlier ones, embodies the best-known features of Z_r . When $r = 0$, α is orthogonal to β; hence its projection onto β is zero. When $r = \pm 1$, $\alpha = \beta$; hence the projection of α onto β has the same length as α. Hyperbolically, that length is infinite. Note that Figures 7 and 8 are not directly comparable for intermediate values

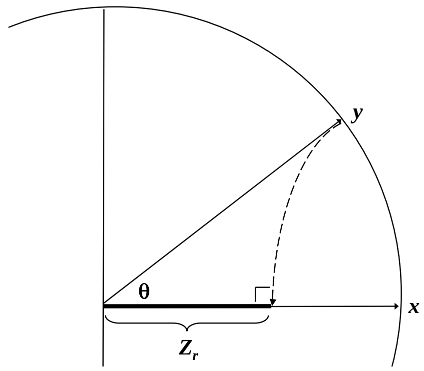

FIGURE 8.
Fisher's Z_r as hyperbolic length

of r, because the two geometric descriptions use different metrics.

The representations we have constructed in Figures 7 and 8 illustrate a geometric property that Z_r shares with r. In Figure 7, r is the Euclidean length of a projection of $\mathbf{y}$ onto $\mathbf{x}$. In Figure 8, Z_r is the hyperbolic projection of $\mathbf{y}$ onto $\mathbf{x}$. Implicit in Figure 7 are some additional features of r. In Figure 7, the distance between $\mathbf{y}$ and its projection onto $\mathbf{x}$ is $\sin\theta$, a natural measure of the lack-of-fit between $\mathbf{y}$ and $\mathbf{x}$ because the vector $P_{\mathbf{x}}(\mathbf{y}) - \mathbf{y}$ is shorter than any other curve connecting the end of $\mathbf{y}$ to a point along $\mathbf{x}$. Thus, r is not merely the signed length of the perpendicular projection of $\mathbf{y}$ onto $\mathbf{x}$. It is the position of the point along $\mathbf{x}$ that is closest to $\mathbf{y}$. Unfortunately, Figure 8 affords no similar interpretation. In fact, the geodesic in Figure 8 that joins $\alpha(\infty)$ (the endpoint of $\mathbf{y}$) with β (the geodesic containing $\mathbf{x}$) has infinite hyperbolic length. Of the many geodesics through $\alpha(\infty)$ that would intersect β, this particular one was chosen because it is orthogonal to β. This geodesic is not, however, any shorter (or longer) than competitors that would have intersected β at a different point – because all such geodesics have infinite hyperbolic length. Hence, Z_r cannot be viewed as the length of a point along β that is closer to $\alpha(\infty)$ than any other point. Nor does Figure 8 provides us with a meaningful measure of the lack of fit between α and β. These interpretations will require a new definition of distance between a point and a geodesic — a definition that we offer in Section 3.3.

3.3. Error minimized by Z_r

The Pearson product-moment correlation coefficient is the least-squares estimator of linear relationship between standardized variables. If these variables are the vectors $\mathbf{y}$ and $\mathbf{x}$ of Section 3.2 above, the least-squares property of the correlation coefficient can be expressed as $|\mathbf{y} - r\mathbf{x}| < |\mathbf{y} - b\mathbf{x}|$ for every $b \neq r$. Textbook authors often use this error criterion to motivate the choice of Pearson's product-moment correlation coefficient as a measure of linear relationship. We now seek an error criterion that would motivate the choice of Fisher's Z statistic. Mathematically, this will be a "distance" function in hyperbolic N-space which is minimized at the value Z_r .

Given a geodesic ray L starting at a point (say the origin) in hyperbolic N-dimensional space and point p not on L, we define the *asymptotic distance* $D(L, p, \infty)$ from L to p to be

$$D(L, p, \infty) = \lim_{t \to \infty} \left(\exp\left(d\left(L(t), p\right) - t\right)\right),$$

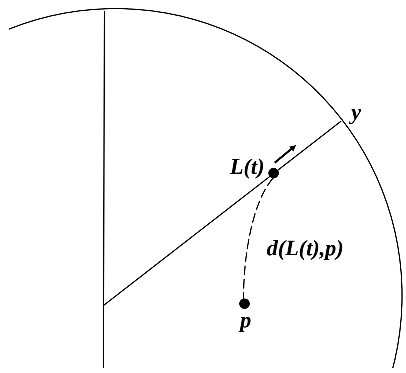

FIGURE 9.
Asymptotic distance criterion

where d denotes hyperbolic distance and $L(t)$ is the point of L that is t (hyperbolic) units from the origin. See Figure 9. In some sense, this measures the hyperbolic proximity of the point p to the end of the geodesic L. To the geodesic triangle whose vertices are the origin, $L(t)$, and p, we apply the hyperbolic Law of Cosines; see Anderson (1999, section 5.7). If θ is the angle between L and the geodesic connecting p with the origin, then

$$\begin{aligned}&\cosh\left(d\left(L\left(t\right),p\right)\right)\\ &=\cosh\left(t\right)\cosh\left(d\left(0,p\right)\right)-\sinh\left(t\right)\sinh\left(d\left(0,p\right)\right)\cos\theta,\end{aligned}$$

so that

$$\begin{aligned}D\left(L,p,\infty\right)&=\lim_{t\to\infty}\left(\exp\left(d\left(L\left(t\right),p\right)-t\right)\right)\\ &=\lim_{t\to\infty}\exp(\operatorname{arccosh}(\cosh\left(t\right)\cosh\left(d\left(0,p\right)\right)\\ &\qquad -\sinh\left(t\right)\sinh\left(d\left(0,p\right)\right)\cos\theta)-t).\end{aligned}$$

Using the estimates $\cosh\left(t\right)=\frac{e^t}{2}\left(1+O\left(e^{-2t}\right)\right)$, $\sinh\left(t\right)=\frac{e^t}{2}\left(1+O\left(e^{-2t}\right)\right)$, $\operatorname{arccosh}\left(x\right)=\log\left(2x\right)+O\left(\frac{1}{x^2}\right)$, we obtain

$$\begin{aligned}D\left(L,p,\infty\right)&=\lim_{t\to\infty}\left(e^t\left(\cosh\left(d\left(0,p\right)\right)-\sinh\left(d\left(0,p\right)\right)\cos\theta\right)e^{-t}\right)\\ &=\cosh\left(d\left(0,p\right)\right)-\sinh\left(d\left(0,p\right)\right)\cos\theta\end{aligned}$$

Returning to the statistics problem, suppose that we have normalized $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbf{R}^N$, and let θ be the angle between the vectors. Let $L_{\mathbf{y}}$ denote a geodesic ray with initial velocity $\mathbf{y}$ at the origin, say, and let $b\mathbf{x}$ denote the point in hyperbolic space that is b units away from the origin in direction $\mathbf{x}$. Suppose that we wish to find b such that $D\left(L_{\mathbf{y}},b\mathbf{x},\infty\right)$ is minimum. Then

$$D\left(L_{\mathbf{y}},b\mathbf{x},\infty\right)=\cosh\left(b\right)-\sinh\left(b\right)\cos\theta,$$

and $\frac{\partial}{\partial b}D\left(L_{\mathbf{y}},b\mathbf{x},\infty\right)=0$ implies that

$$\begin{aligned}0&=\frac{\partial}{\partial b}\left(\cosh\left(b\right)-\sinh\left(b\right)\cos\theta\right)\\ &=\sinh(b)-\cosh(b)\cos\theta,\end{aligned}$$

or

$$b = \operatorname{arctanh}(\cos\theta) = Z_r\,,$$

which implies that

$$\begin{aligned} D\left(L_{\mathbf{y}}, Z_r\mathbf{x}, \infty\right) &= \cosh\left(Z_r\right) - \sinh\left(Z_r\right)\tanh\left(Z_r\right) \\ &= \frac{\cosh^2\left(Z_r\right) - \sinh^2\left(Z_r\right)}{\cosh\left(Z_r\right)} \\ &= \frac{1}{\cosh\left(Z_r\right)} = \frac{1}{\cosh\left(\operatorname{arctanh}\left(\cos\theta\right)\right)} \\ &= \sqrt{1-\cos^2\left(\theta\right)} = \sin\theta. \end{aligned}$$

We check the second derivative:

$$\begin{aligned} \frac{\partial^2}{\partial b^2}\left(\cosh\left(b\right) - \sinh\left(b\right)\cos\theta\right) &= \cosh\left(b\right) - \sinh\left(b\right)\cos\theta \\ &= D\left(L_{\mathbf{y}}, b\mathbf{x}, \infty\right) > 0, \end{aligned}$$

to find that the asymptotic distance $D\left(L_{\mathbf{y}}, b\mathbf{x}, \infty\right)$ in fact achieves a global minimum value of $\sin\theta$ at $b = Z_r$. Observe that this newly-defined asymptotic distance is the same as the minimum Euclidean distance from $\mathbf{y}$ to $b\mathbf{x}$, if $\mathbf{y}$ and $\mathbf{x}$ have Euclidean length 1.

4. Conclusion

Here we have developed the first geometric interpretations of Fisher's Z_r transformation. As our work reveals, Z_r is geometrically similar to r; indeed, the similarities are so strong that we regard Z_r as the hyperbolic counterpart to the Euclidean r. Our constructions illustrate well-known features of these two statistics and allow us to see the r-to-Z_r transformation for the first time.

The geometric context of this paper suggests many additional questions such as the following, which have not yet been considered. Can the sampling properties of the Z_r statistic be understood in a geometric way? Does every transformation of r suggest a particular type of geometry? We offer the present work with the hope of inspiring additional insights.

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