Information and Analyticity

M. A. Fiddy

Center for Optoelectronics and Optical Communications

University of North Carolina, Charlotte

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Hari Allamsetty and Umer Shahid

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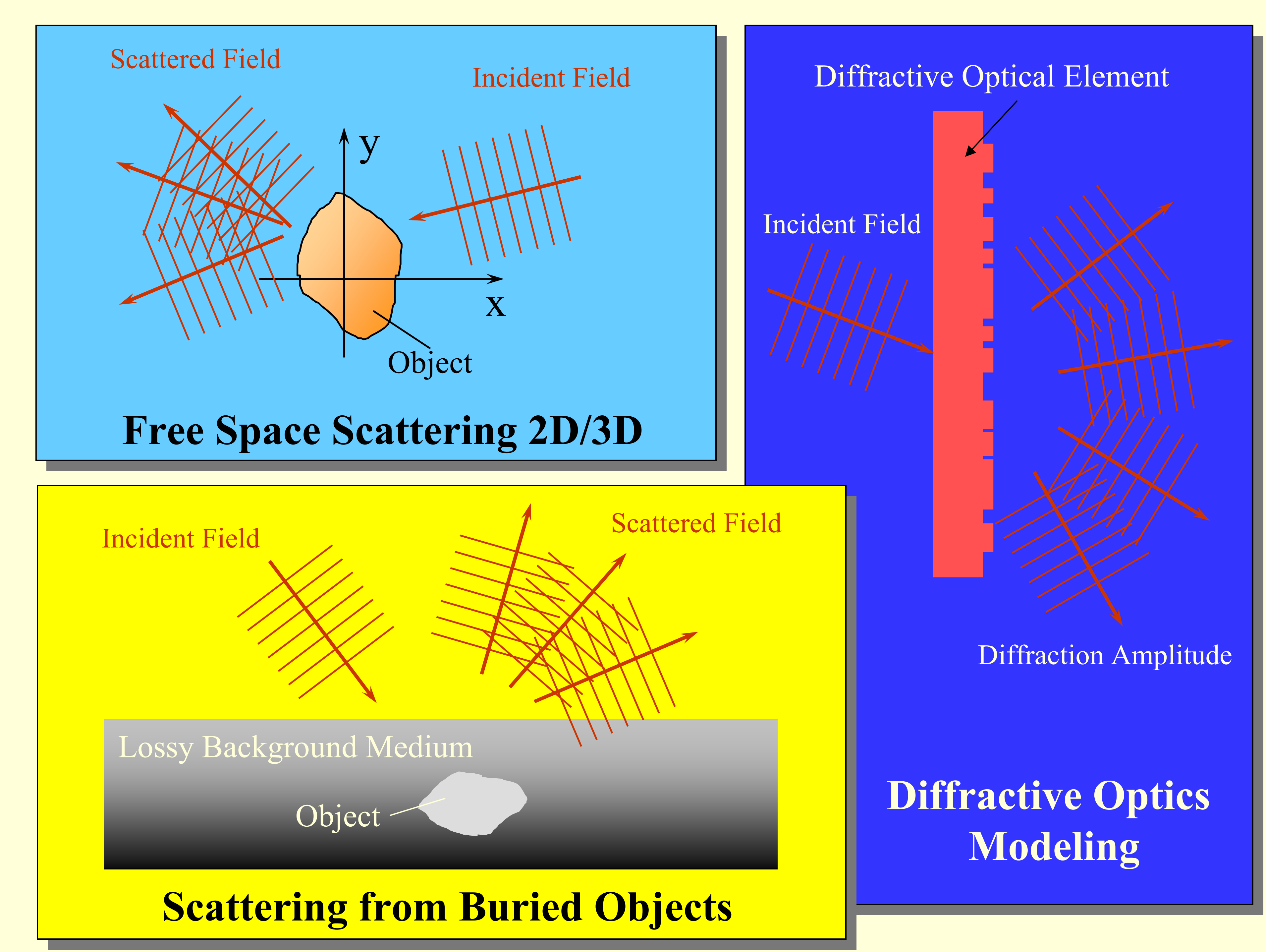
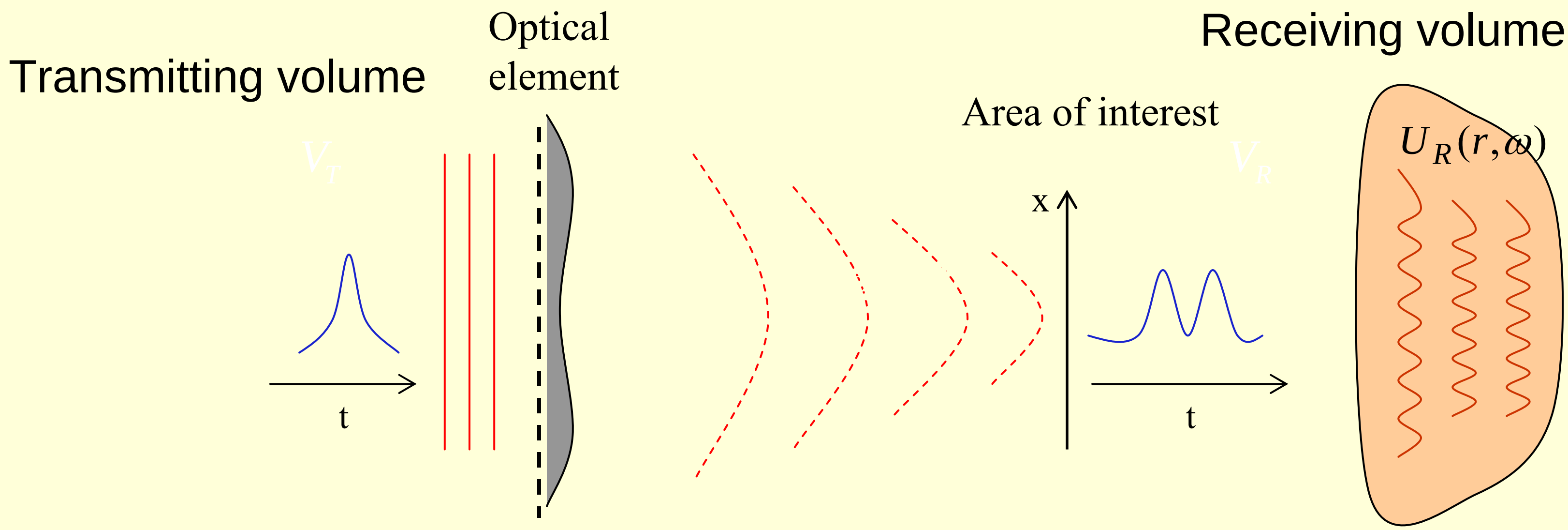
"*The Physical Nature of Information*", May 12th 2004



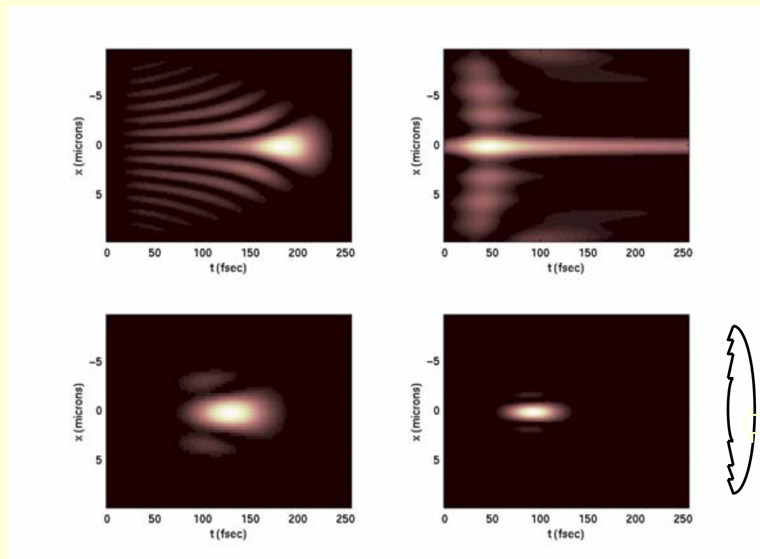
Scattered Field
Incident Field
y
x
Object
Free Space Scattering 2D/3D
Diffractive Optical Element
Incident Field
Diffraction Amplitude
Diffractive Optics Modeling
Incident Field
Scattered Field
Lossy Background Medium
Object
Scattering from Buried Objects

Spatio-temporal pulse shaping problem



$$U_R(r,\omega) = U^{\text{inc}}(r,\omega) + \int_{V_T} G(r,r',\omega)\psi(r',\omega)dr'$$

$G(r,r',\omega)$: Green's function



Piestun

Motivation

There are many imaging methods, inverse problems and synthesis/design problems for which some needed information is missing and difficult to estimate.

Sampling strategies/theorems

Superresolution

Phase retrieval

Inverse scattering problems

Analytic properties of waves and many physically important functions can provide insights, constraints and solutions to these problems.

Causality and Dispersion Relations

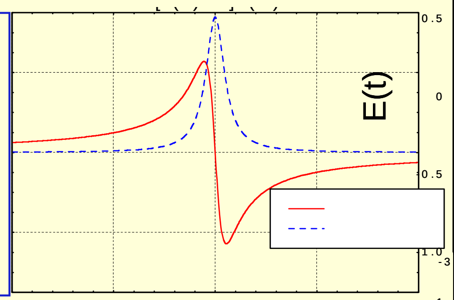
Mathematical statement of causality:

$$E = 0, \quad t < 0$$ i.e. no response before E applied

$$\Rightarrow \chi(t) = 0 \text{ for } t < 0$$

$$\chi'(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi''(\omega')}{\omega-\omega'}d\omega'$$

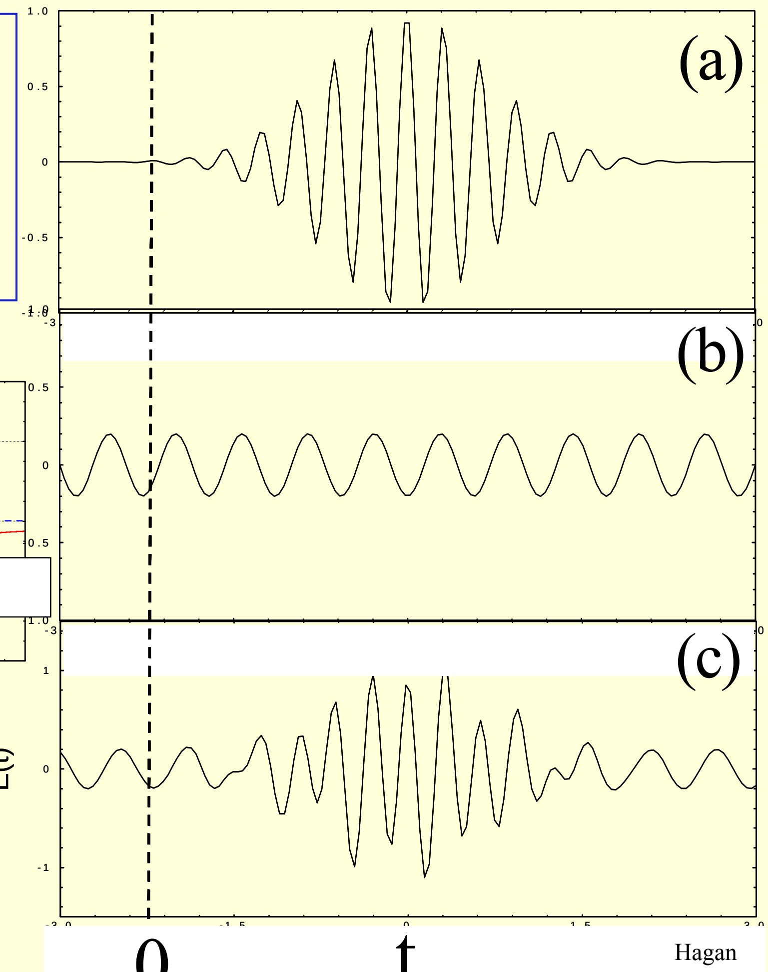
$$\chi''(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi'(\omega')}{\omega-\omega'}d\omega'$$



Titchmarsh's Theorem p125, *Intro to the Theory of Fourier Integrals,* 2nd Ed, OUP, 1948

J. S. Toll Phys.Rev.104(1956) pp1760-1770.

The famous Kramers-Kronig relations in optics have been known since 1927



The subtraction of a single Fourier component from the input pulse field is shown in part (c) of the figure. It no longer obeys causality: analyticity.

Blackbody Radiation

PROC. PHYS. SOC., 1962, VOL. 80

Is a Complete Determination of the Energy Spectrum of Light Possible from Measurements of the Degree of Coherence?

By E. WOLF

Department of Physics and Astronomy, University of Rochester, Rochester, New York

MS. received 30th April 1962

PROC. PHYS. SOC., 1962, VOL. 80

Temporal Coherence of Black Body Radiation

By Y. KANO and E. WOLF

Department of Physics and Astronomy, University of Rochester, Rochester, New York

MS. received 30th April 1962

Problem:

Recover spectral information g(ν) from modulus of the complex degree of coherence |γ(τ)|

{Hanbury Brown and Twiss}

Solution:

Consider analytic properties of γ(τ)

Since g(ν) is causal, invoke Titchmarsh's theorem which allows a dispersion relation (Hilbert transform) to be written between Re{γ(τ)} and Im{γ(τ)} or Re{log γ(τ)} and Im{log γ(τ)} under certain conditions.

Recovering missing phase information

Phase retrieval

…….assume field analytic function (regular in uhp)

F = |F|exp(iϕ) = FT{f} where f causal or of compact support

$$\phi\,(\;.\;) = \frac{1}{\pi}\oint_{uhp}\left[\frac{\log|F(x)|dx}{x - x'} + \cap\right]$$

“Minimum phase” condition: zero free half plane

Kano and Wolf proved that for *blackbody* radiation,
$\gamma(\nu)$ □ $\tilde{□}$ ζ□ıiτ) / π^4 **and** has a zero-free half plane.

ζ : **generalized Riemann zeta-function**

Question posed (p1271): seems likely that zeros in this half plane should have physical significance?

Sampling methods

The theorem was first formulated by Harry Nyquist in 1928 ("Certain topics in telegraph transmission theory"), but was only formally proved by Claude E. Shannon in 1949 ("Communication in the presence of noise")

$$F(s) = \mathcal{F}(f)(s) = \int_{-\infty}^{\infty} f(t)e^{-its}\,dt.$$

$$f(t) = \mathcal{F}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{its}\,ds.$$

If a function *s(x)* has a Fourier transform ***F[s(x)] = S(f) = 0* for *|f| > W***, then it is **completely determined** by giving the value of the function at a series of points spaced *1/(2W)* apart. The values s_n = *s(n/(2W))* are the *samples of s(x)*.

The Nyquist-Shannon Sampling Theorem states that if a function *s*(*x*) has a Fourier transform *F*[*s*(*x*)] = *S*(*f*) = 0 for | *f* | > *W*, then *s*(*x*) can be recovered from its samples s_n by the formula:

$$s(x) = \sum_{n=-\infty}^{\infty} s_n \frac{\sin\left(\pi(2Wx-n)\right)}{\pi(2Wx-n)}$$

Infinite number of uniformly distributed samples required: interpolation function is zero at all sample points but one and **asymptotic zeros at 1/2W**.

Nonuniform samples (e.g. Yen 1956): replace sinc(.) by $L_n(x) = G(x)/[(x - z_n)\, G^1(z)]$ where $G(z)=\prod(1-z/z_n)$ and $L_n(x_k) = \delta_{nk}$ giving $s(x) = \sum F(x_n)L_n(x)$ **provided asymptotic zeros** spaced at intervals of 1/2W.

Sampling methods

For *f(u) = 0* for |*u*| > *W* (*u* = space, time, frequency…) then F(x) can be represented by

i) its samples $F(x_n)$ **or**

ii) its roots or zeros using the Hadamard product.

$$F(x) = \int_{\square} f(u)\exp(-ikux)du \sim \prod_{j=-\infty}^{\infty}(1-\tilde{\square}\square\square)$$

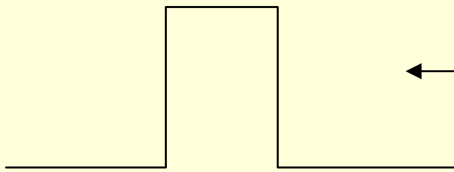
FT of function with compact support is a W-bandlimited function, or entire function of exponential type W.

Its growth properties and zero distributions are highly constrained.

It can be encoded by samples at 1/2W or by its zero locations, much like a polynomial. The zeros of the function have some physical significance.

Nonuniform sampling is possible as is slower than Nyquist sampling rates, i.e. at m/2W (see Papoulis IEEE CAS-24, 652, (1977))

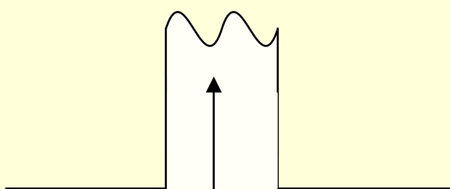
How do zeros code information?



FT

$$\sin(k\Delta u.x) = \prod_{j\neq 0}\left[1 - x/\{j\pi/k\Delta u)\}\right]$$

Δu



Moving mth zero in x-domain introduces m cycles in u-domain: *amplitude* ~ distance from $m\pi/k\Delta u$ and *phase* ~ arctan(imag co-ord/distance).

[Fiddy et al, Opt Acta 29, (1982), p23-40]

λ/2 limit?

Space-bandwidth product ~ $\Delta x \Delta u$

[Lukosz JOSA 57 p932, (1967)]

Information Retrieval Problems

Missing phase

Limited data

Multiple scattering


PROCEEDINGS OF THE IEEE, VOL. 69, NO. 5, MAY 1981 529

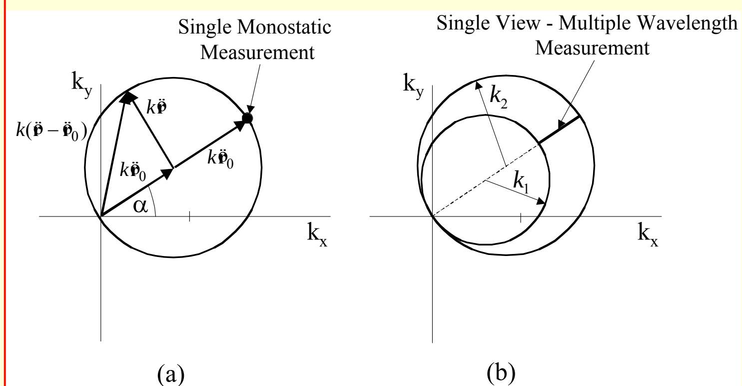
The Importance of Phase in Signals

ALAN V. OPPENHEIM, FELLOW, IEEE, AND JAE S. LIM, MEMBER, IEEE

(a) (b)

(c) (d)

Fig. 3. (a) Original image A. (b) Original image B. (c) Image synthesized from the Fourier transform phase of image A and the magnitude of image B. (d) Image synthesized from the Fourier transform magnitude of image A and the phase of image B.



$$\psi_s(\mathbf{r}, k\hat{\mathbf{r}}_o) = k^2 \frac{e^{ikr}}{4\pi r}\int_D d\mathbf{r}' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} V(\mathbf{r}')\Psi(\mathbf{r}', k\hat{\mathbf{r}}_o)$$

Recovering missing phase information

Phase retrieval

…….assume field analytic function

F = |F|exp(iφ) = FT{f} where f causal and/or of compact support

$$\phi(\;.\;) = \frac{1}{\pi}\oint_{uhp}\left[\frac{\log|F(x)|dx}{x - x'} + \cap\right]$$

"Minimum phase" condition: zero free half plane

Zero flipping and phase ambiguities

1D problems:

$$F(x) = \int_{\Delta} f(u)\exp(-ikux)du \sim \prod_{\square=-\infty}^{\infty}(1-\tilde{\square}\square\square)$$

□□□□□□□□□ I ~ F.F* *we need to factor* I

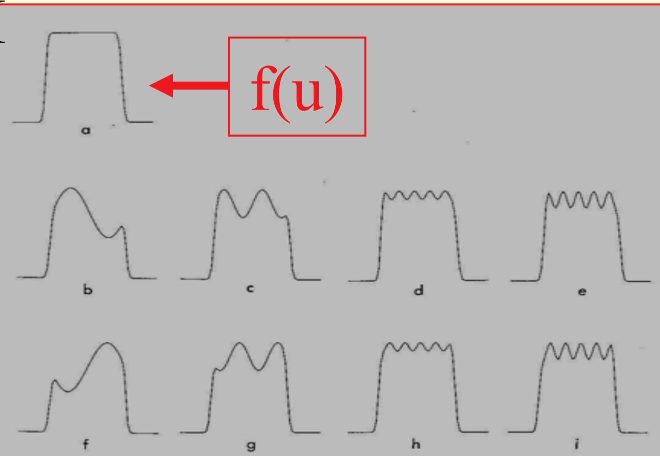


Figure 1. (*a*) F. T. of fundamental sinc zero lattice; (*b*), (*f*) F.T. after movement of z_1 from $y_1 = 0$ to $y_1 = +k, -k$ respectively; (*c*), (*g*) as (*b*), (*f*) but y_2 moved; (*d*), (*h*) as (*b*), (*f*) but y_5 moved; (*e*), (*i*) as (*d*), (*h*) but y_5 moved ± 2k.

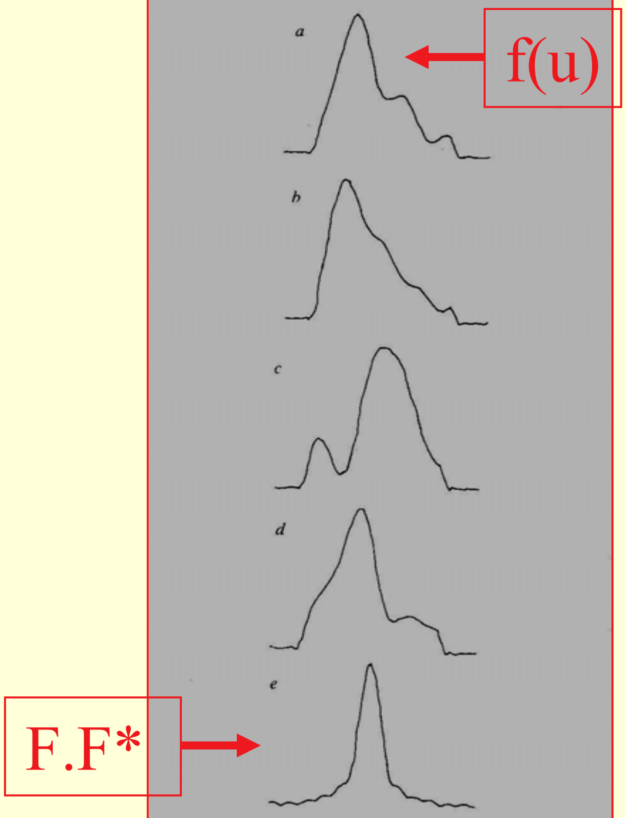
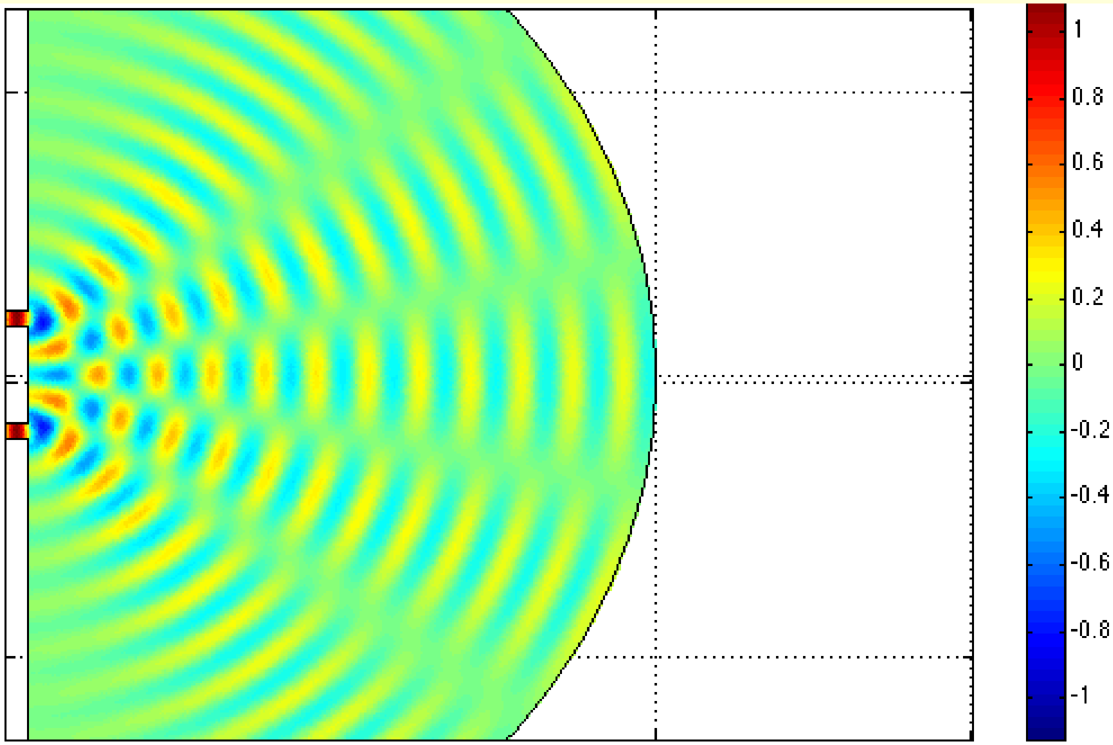


Fig. 1 All four different real positive objects shown in *a*, *b*, *c* and *d* produce the same far field intensity, shown in *e*. The four different objects have been generated by reflecting or 'flipping' the zeros of $F(z)$ about the *x*-axis, as described in the text.

Early work: **O'Neill and Walther, Opt. Act. 10, p33, 1963; Kohler and Mandel, J.O.S.A. 63, p126, 1973.**

Zeros and their trajectories in diffracted fields



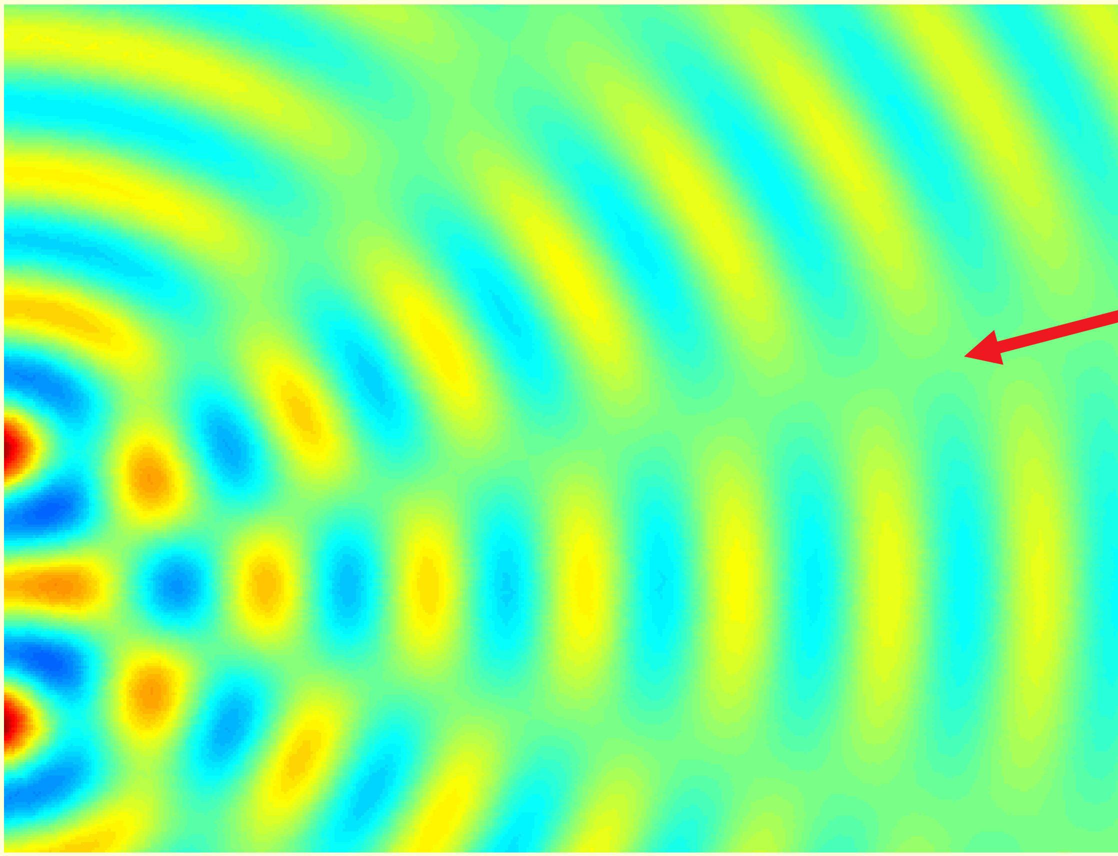
Schouten, Visser and Wolf, *New effects in Young's interference experiment with partially coherent light,* Opt. Lett., 28, p1182, 2003

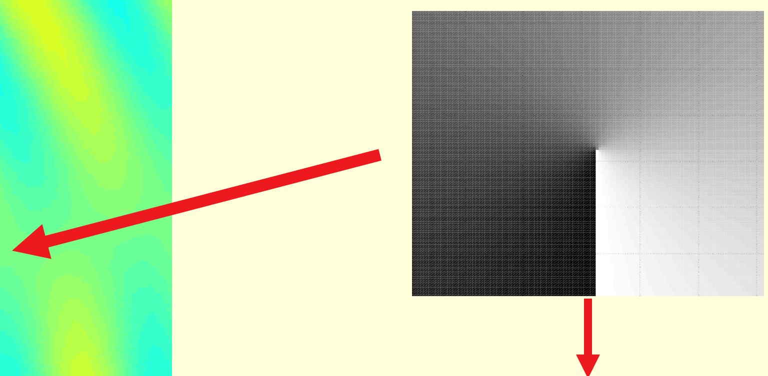
Winthrop/Gabor:
"tubes" of information

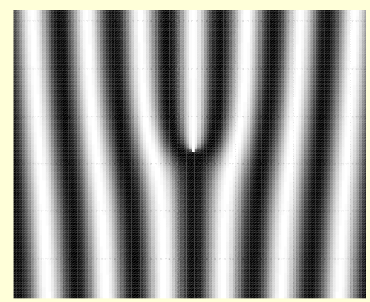
The study of the zero trajectories of diffraction patterns has lead to the relatively new field of "singular optics"

e.g. Soskin and Vasnetsov, ch 4, Prog. in Optics, 42, 2001, ed. Emil Wolf.

Phase dislocation at zero in field



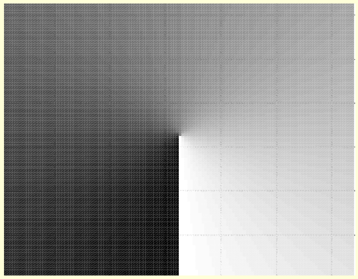


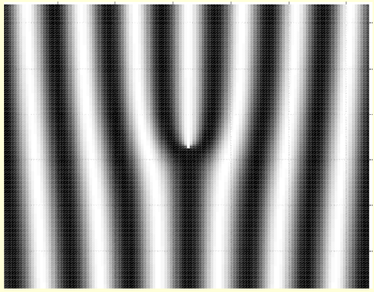


Zeros and vortices

The zeros of an analytic function encode that function just like (Shannon) sampling points…….and

around each zero point in the intensity of a field, there is a phase discontinuity of 2π. and a spiral phase structure or a wavefront dislocation.





Local analytic representation is

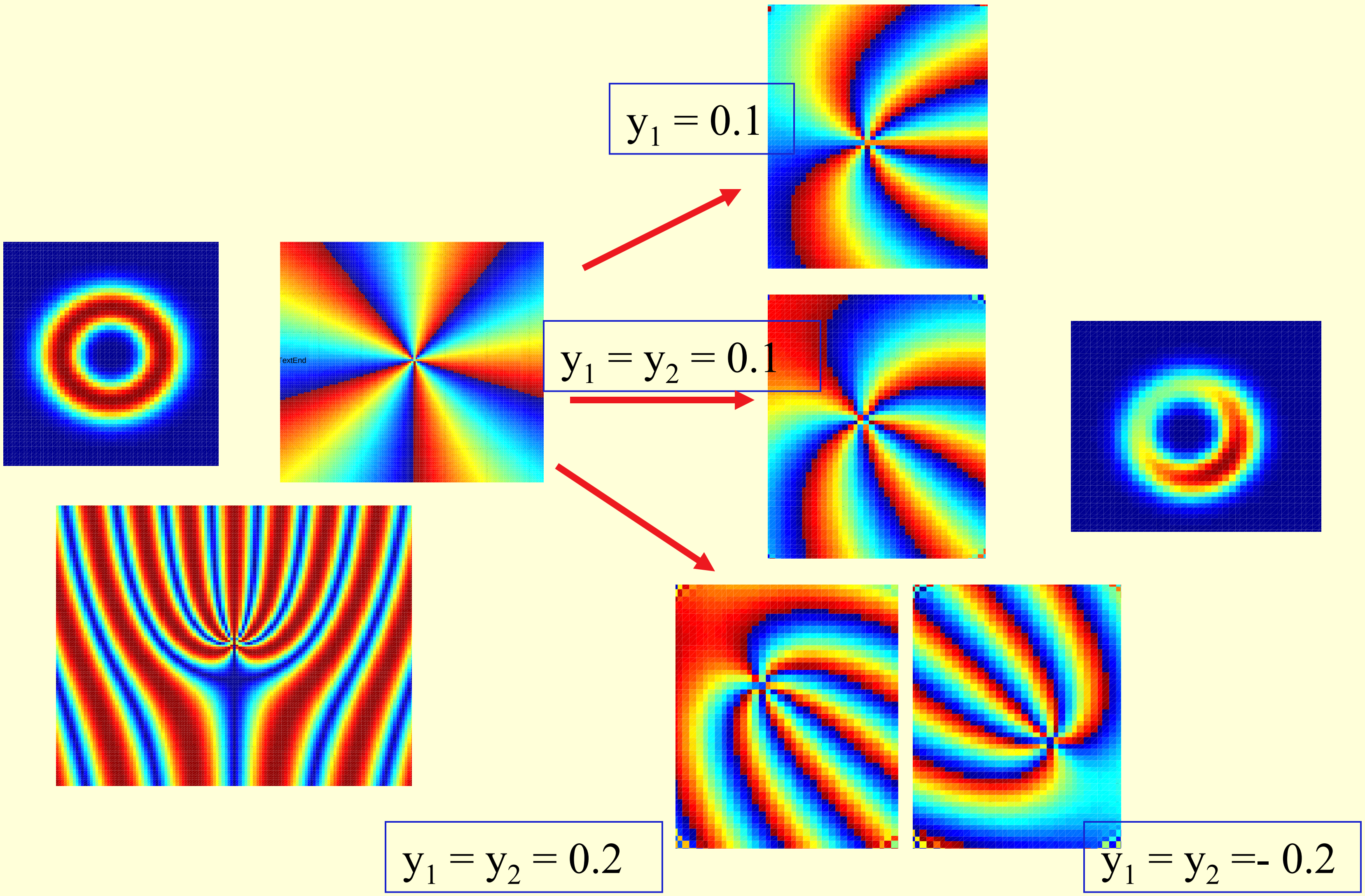
$$(x_1 + ix_2)^n$$

$$(z_1 + iz_2)^n$$

$$z = x + iy$$

$$F(z_1,z_2) = F(x_1 + iy_1, x_2 + iy_2) = \int f(u,v)\exp(-y_1u - y_2v)\, \exp\{i2\pi\, (x_1u + x_2v)\}\, dudv$$

Properties of a fifth-order vortex



Spectral anomalies at wavefront dislocations

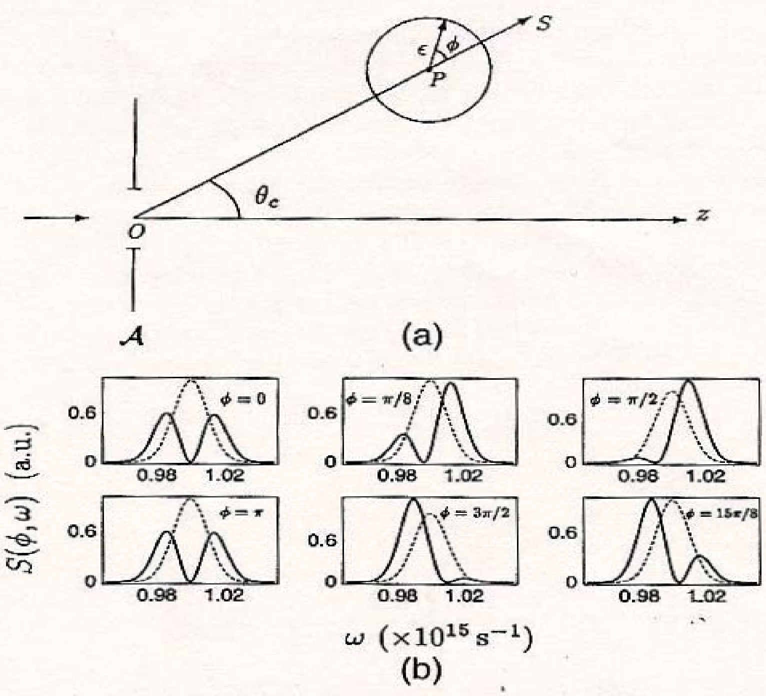
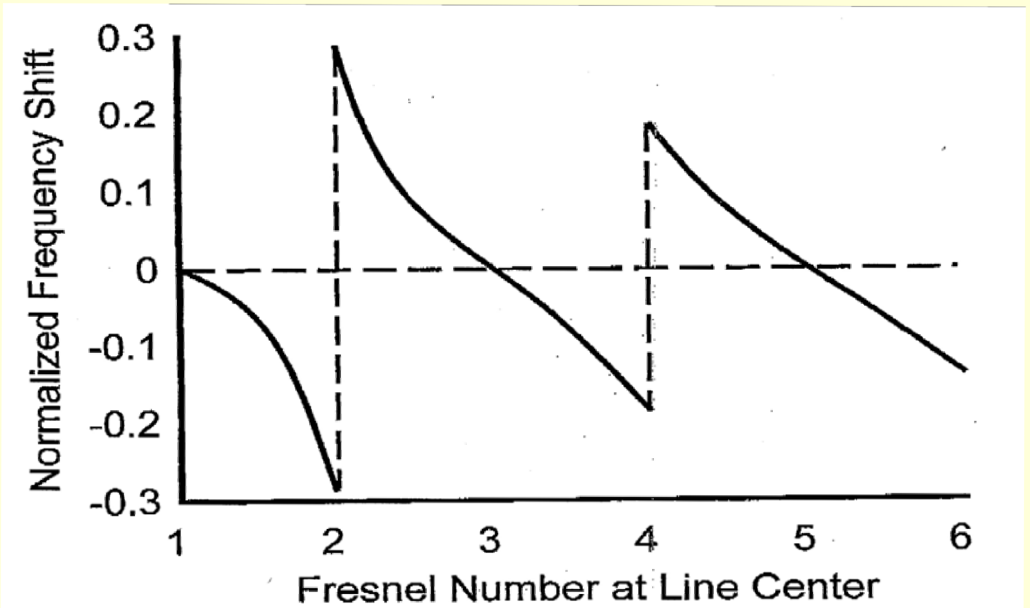


Fig. 5. Spectral changes in the neighborhood of a zero at a point P in the Airy diffraction pattern at frequency ω_0 of a circular aperture, along a small circular loop of radius ϵ centered at P, which is at the distance r from the center O of the aperture. The vector OS points toward the critical direction that makes the angle θ_c with the z axis. The spectrum is displayed for different values of the angle ϕ. The numerical parameters are chosen so that $\epsilon / r\theta_c = 0.005$.

S. A. Ponomarenko and E. Wolf, *Spectral anomalies in a Fraunhofer diffraction pattern*, Opt. Lett., July 2002, 27, p1211.

Spectral anomalies at wavefront dislocations



Diffraction induced abrupt spectral red and blue shifts at critical distances where the dominant spectral component of the diffracted field has a zero.

J. Foley and E. Wolf, *Phenomenon of spectral switches as a new effect in singular optics with polychromatic light*, JOSA A, 19, p2510, 2002

see also Pu and Nemoto, Spectral changes and 1 X N spectral switches, JOSA A 19, p339, 2002

Vortices for superresolved lithography

PHOTOMASK

BACUS—The international technical group of SPIE dedicated to the advancement of photomask technology.

BACUS NEWS
APRIL 2004
VOLUME 20, ISSUE 4

Vortex Via Validation

Marc D. Levenson, M.D. Levenson Consulting, Saratoga CA 95070
Takeaki (Joe) Ebihara, Canon USA Inc., San Jose, CA 95134
Yasutaka Morikawac & Naoya Hayashic, Dai Nippon Printing Co., Ltd., Kamifukuoka, Japan

ABSTRACT

The first vortex masks composed of rectangles with phases of 0°, 90°, 180°, and 270°—as proposed at Photomask 2002—have been fabricated and shown to print sub-100nm contacts. The walls of the phase trenches are very nearly vertical, with all four phase regions meeting at sharp corners, which define the phase singularities. Arrays with pitches down to 210nm have been printed in negative DUV resist using KrF illumination with NA=0.73 and σ=0.15. The developed contacts are somewhat elliptical, but their shapes can be corrected (if necessary) by OPC techniques. The depth of focus for ±10% CD variation is ~400nm for 85nm CD vias at 210nm pitch and ~500nm for 95nm vias at 250nm pitch - with 12% exposure latitude. At constant exposure dose, the via CDs vary with pitch approximately as predicted by simulations. Increasing exposure dose makes the openings smaller, more uniform and more circular. No significant surface development has appeared due to phase-edge printing. However, the spacewidth alternation

Continues on page 4.

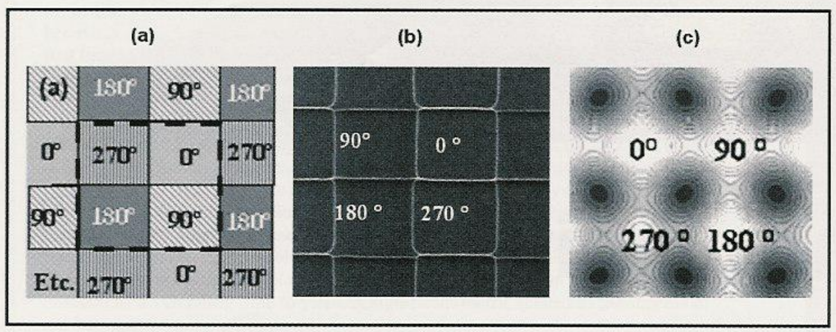
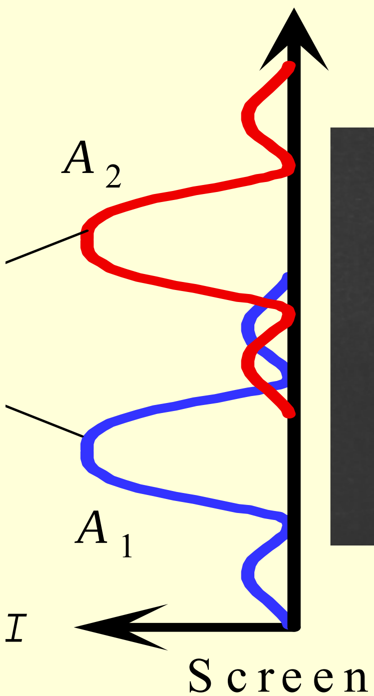
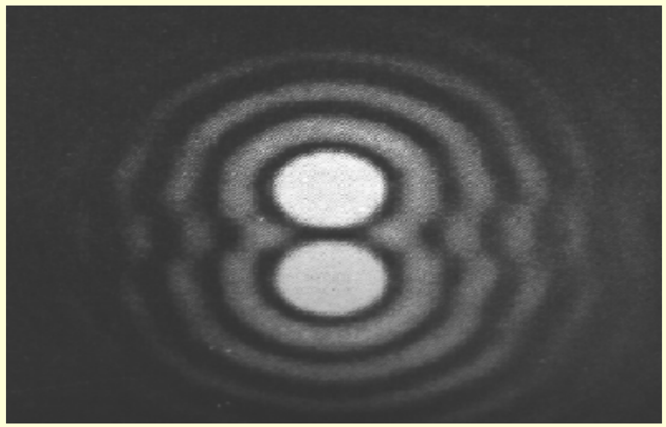


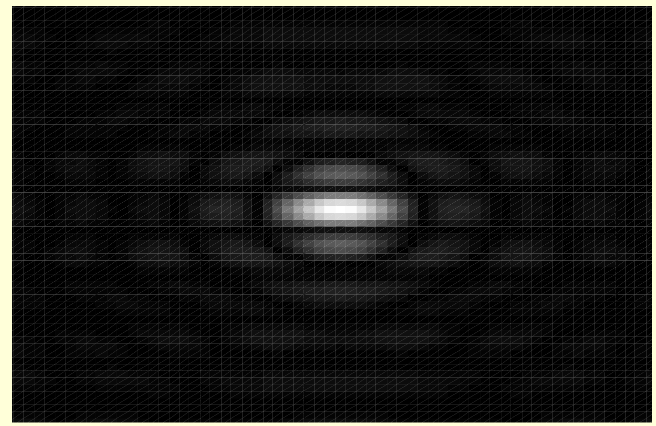
Figure 1. Typical N=4 chromeless vortex via mask array design (a) along with a top-down SEM image of such a mask (b). Dashed lines enclose the repeating unit cell. Figure 1c shows the expected aerial image for a 250nm pitch array. Note that the maximum intensity of the simulated image is above the flood exposure level I_∞.

TAKE A LOOK INSIDE!

Zeros and superresolution







Separable and hence reducible analytic functions only in ideal simple noise free cases

E. Wolf and M. Nieto-Vesperinas, "*Analyticity of the angular spectrum amplitude of scattered fields and some of its consequences*", J.O.S.A. A2, p886, 1985

Definitions

Strehl ratio = S =

intensity of superres peak
intensity of Airy disk

G = 1st zero of superres peak
______1st zero of Airy disk

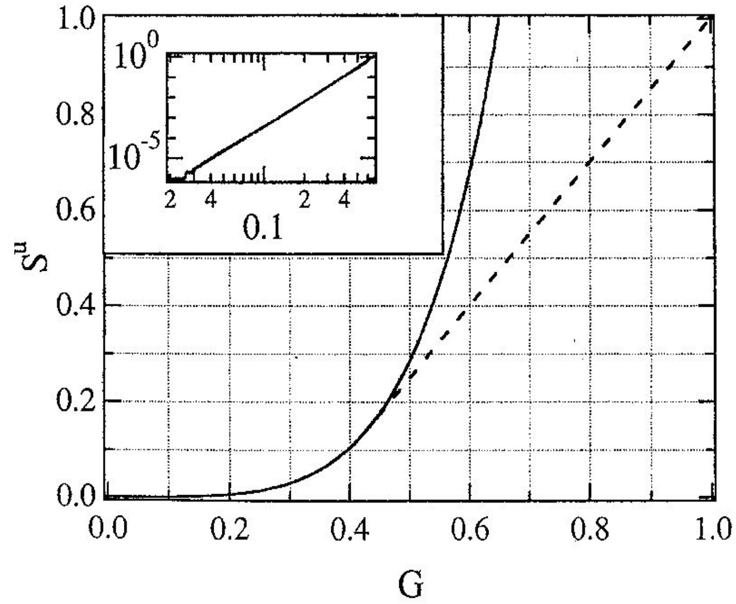


Fig. 2. Upper bound for the Strehl ratio S^u when conventional imaging is used. The solid curve corresponds to the solution obtained with the series expansion for S^u, and the inset shows the same data on a logarithmic scale. The dashed curve is a better estimate of the upper bound in the region $G \in (0.46\ 1]$.

Tasso and Morris, *Fundamental limits of optical superresolution*, OL 22, p582, 1997

Synthesis of Superresolving Filters

Fraunhofer diffraction

$$F(x,y) = \iint f(u,v)\exp(-ik(ux+vy))\,dudv$$

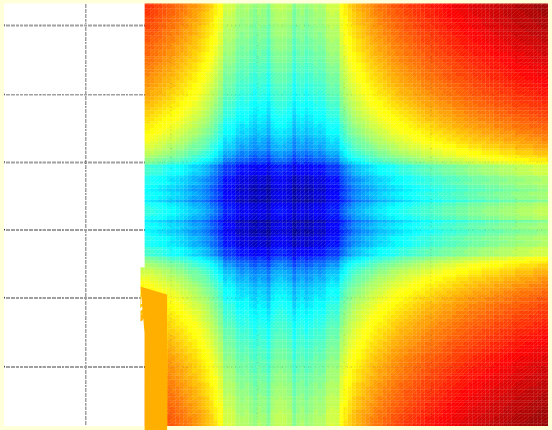
Far-field scattering integral

$$F(x,y) = \iint f(u,v)F_T(u,v)\exp(-ik(ux+vy))\,dudv$$

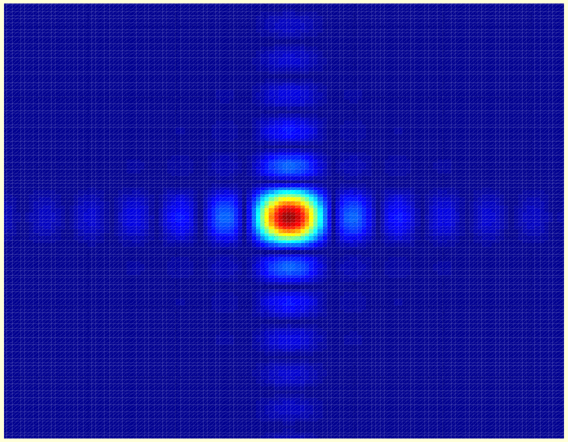
where $F_T = \Psi/\Psi_o$ and $\Psi = \Psi_o$ in 1st Born approximation

……and $F(x,y)$ is an entire function.

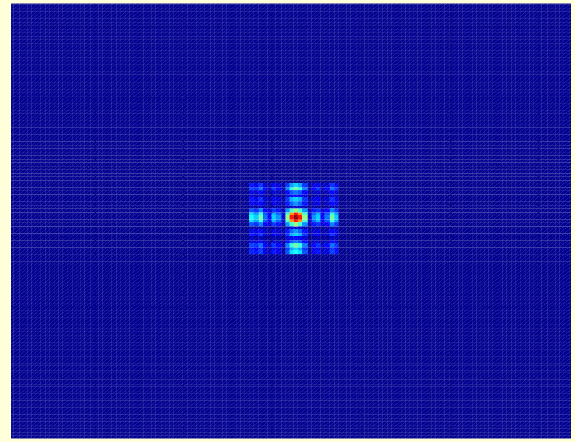
Results suggest that methods attempting to invert <u>multiply scattered</u> data do permit some sub-λ superresolution (e.g Chen and Chew, App.Phys.Lett, 72, p1284, (1998); *multiple scattering between sub-structures leading to evanescent waves which couple into propagating waves……a degree of freedom for synthesis problems*)



Superresolved PSF from filter

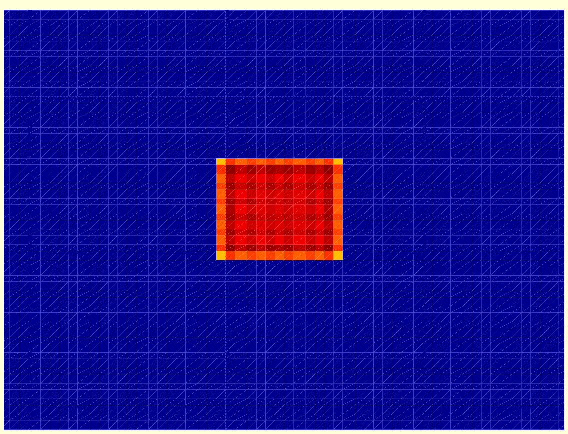


PSF from aperture

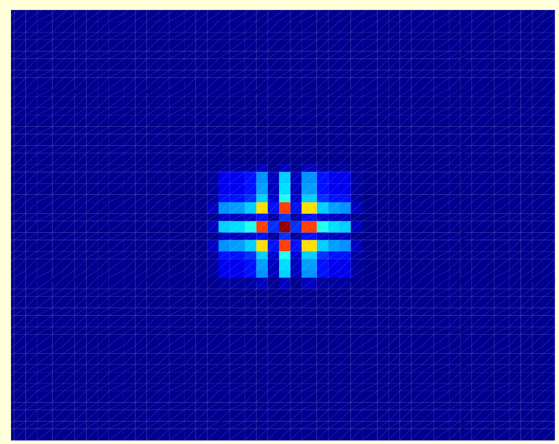


PSF from superresolved filter

Superresolving filter



Field at aperture



Field $f(u,v)F_T(u,v)$ exiting filter

2D analytic functions

2D problems:

No fundamental theorem of algebra: generally irreducible

$$F(x,y) = \int_{\Delta} f(u,v)\exp(-ik[ux + vy])\, du.dv \sim \Pi ?$$

□□□□□□□□□ $I \sim F.F^*$ Phase retrieval: *we still need to factor* I

Analytic properties:

i) similar to 1D but no simple concept of zero-free half "plane"

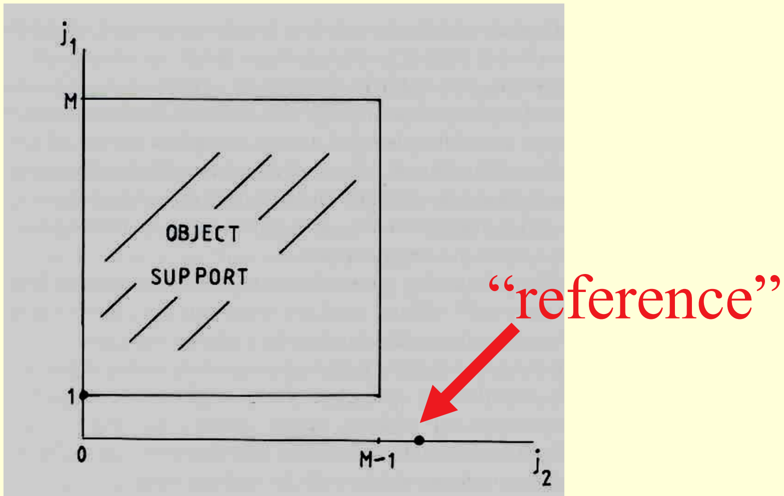
ii) zeros are 2n - 2 dimensional in nD case and cannot be isolated points

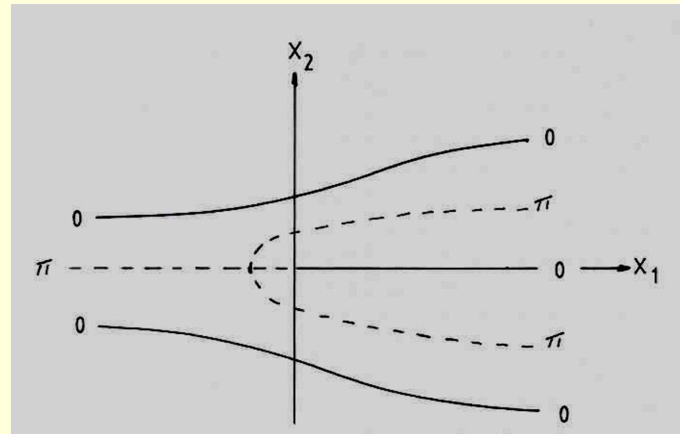
iii) no unique definition of dispersion relations (which is consistent with causality condition)

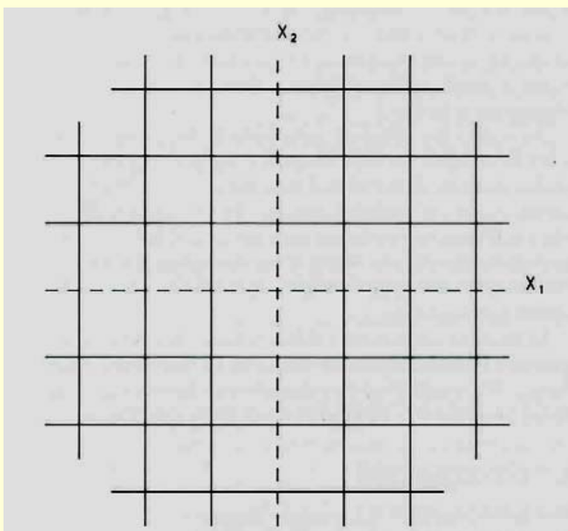
Approximate factoring of 2D analytic functions, F, accomplished by taking logF, and separating additive terms to become factors…..if logF is well behaved.

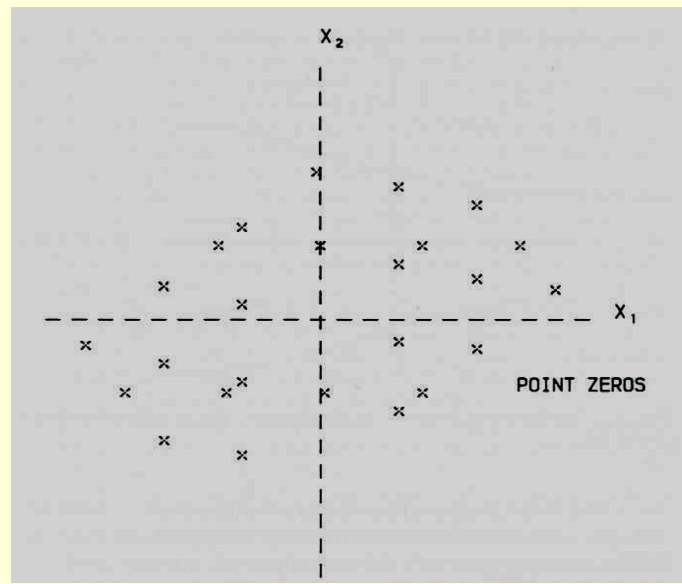
Wiener-Levy Theorem: if a continuous F > 0 has an absolutely convergent Fourier series (FS) then logF has an absolutely convergent FS. [AMS Trans, p791, (1933)]

Effect of a reference wave on phase



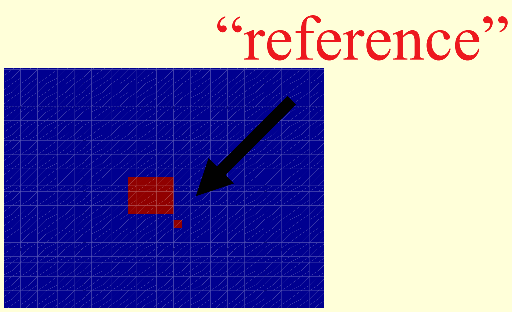


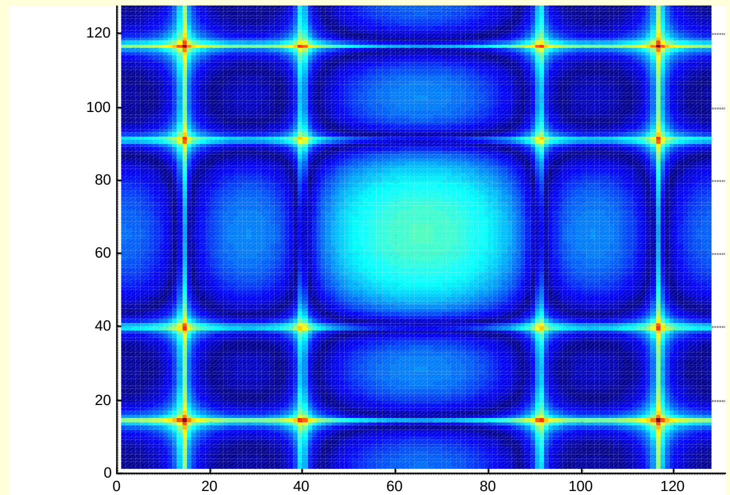


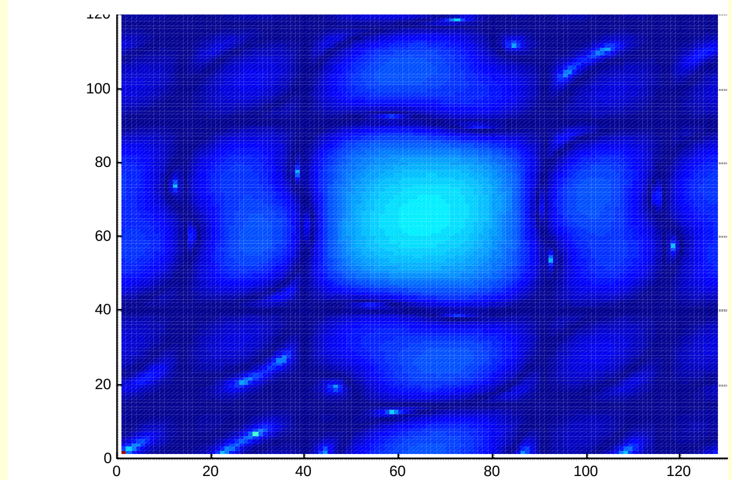


Scivier and Fiddy, JOSA A2 (1985) p693-697

Disrupting the phase of a wavefront

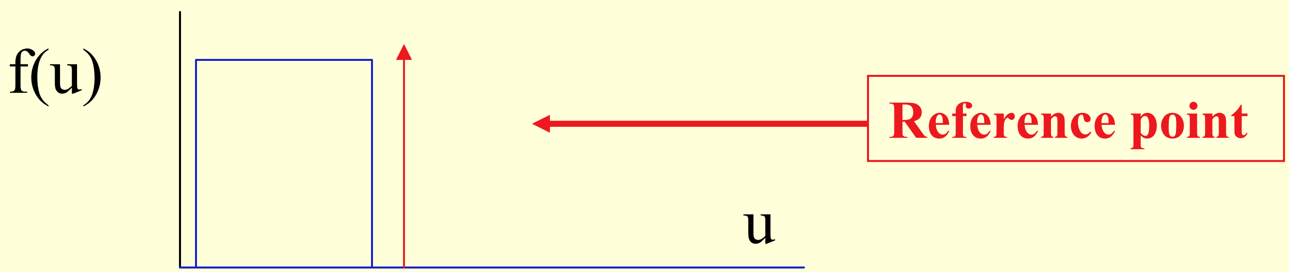






Ensuring the minimum phase condition in 1D

Force minimum phase condition by introducing a reference wave (Rouche’s theorem)



Minimum phase condition:

$$-\pi < \phi \leq \pi$$

□□□.”□□□□□□□□□□.”□□□□□□□□
□**…useful…**

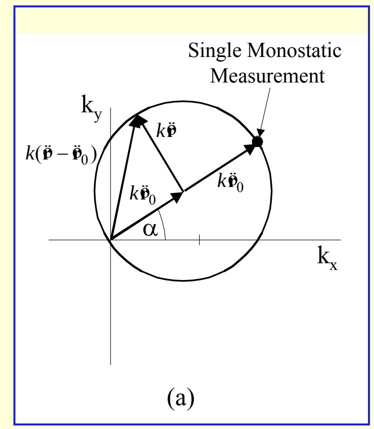
Inverse scattering

______…possible by *nonlinear* filtering

…if have minimum phase function

$$\psi_s(\mathbf{r}, k\hat{\mathbf{r}}_o) = k^2 \frac{e^{ikr}}{4\pi r}\int_D d\mathbf{r}'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\, V(\mathbf{r}')\,\Psi(\mathbf{r}', k\hat{\mathbf{r}}_o)$$

$$= \int_D d\mathbf{r}'\, e^{-ik(\hat{\beta} - \hat{\alpha})\cdot\mathbf{r}'}\; V(\mathbf{r}')\, \frac{\Psi(\mathbf{r}', k\hat{\alpha})}{\Psi_o(\mathbf{r}', k\hat{\alpha})}$$



$$V\frac{\psi}{\psi_0} \rightarrow \ln(V) + \ln\left(\frac{\psi}{\psi_0}\right) \rightarrow FT\left[\ln(V)\right] + \cancel{FT\left[\ln\left(\frac{\psi}{\psi_0}\right)\right]} \rightarrow \ln(V) \rightarrow V'_{est,n}$$

spatial filter

Homomorphic filtering for factoring

$$V_B(\mathbf{r}, k\hat{\mathbf{r}}_o) \;-\; V(\mathbf{r})\,\frac{\Psi(\mathbf{r}, k\hat{\mathbf{r}}_o)}{\Psi_o(\mathbf{r}, k\hat{\mathbf{r}}_o)}\;.$$

$$V\Psi \longrightarrow \log V\Psi = \log|V| + \log|\Psi| + i[\arg(V) + \arg(\Psi)]$$

Inverse FT therefore not amenable to filtering to isolate V from Ψ

Phase contains discontinuities (wavefront dislocations)

Phase unwrapping virtually impossible in $\geq$ 2D

Minimum phase condition

Minimum phase condition in 1D: zero free half plane

Minimum phase condition in >1D:

Dudgeon and Mersereau *[Multidimensional Digital Signal Processing*, Prentice Hall, ch 4, p202*]:*

"Multidimensional minimum phase signal is one that is absolutely summable and whose inverse and complex *cepstrum* have the same region of support"

Definition: **Cepstrum of g is $FT^{-1}\{\log G\}$ where $G = FT\{g\}$**

Creating a 2D minimum phase function

Let cepstrum = g

$FT\{g\} = G = |G| \exp(i\theta) = Re\{G\} + iIm\{G\}$

Create an object:

$\exp(G) = \exp(Re\{G\}).\exp(iIm\{G\})$ [$\equiv F = |F| \exp(i\theta)$]

Hence $\exp(iIm\{G\}) = \exp(i\theta)$ ……**this phase will be unwrapped and should be a "minimum" phase function….**

and $\exp(Re\{G\}) = |F|$ ….**so no zeros!**

Apply Rouche's theorem in 2D.

Trivial case: with a strong reference wave added to F, i.e. 1 + F then ~ exp(F) for |F| << 1 and FT{1 + F} = δ + f; thus we expect to recover f when a (strong) reference point is present somewhere in the object domain.

Creating a 2D minimum phase function

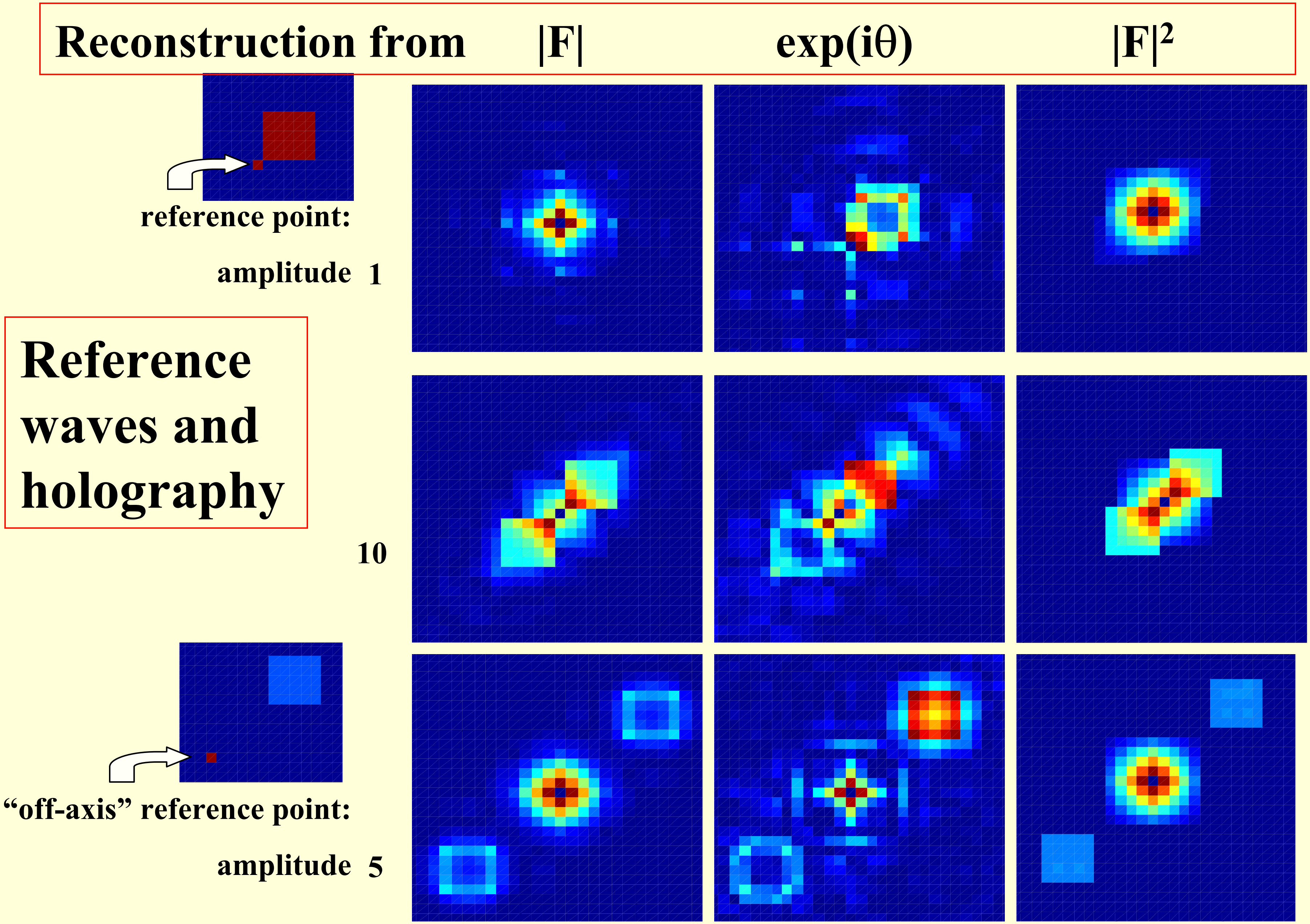
<u>Rouche's Theorem in 1D and nD</u>

Suppose w = f(z) is analytic in a domain D where $f = (f_1, f_2, \ldots f_n)$ and the boundary of D is smooth and contains no zeros of f, then if for each point z on the boundary, there is *at least* one index j (j = 1,2….n) such that $|g_j(z)| < |f_j(z)|$ then g(z) and g(z) + f(z) have the same number of zeros in D.

[It actually suffices that $Re\{g_j(z)\} < Re\{f_j(z)\}$; see Integral Representations and Residues in Multidimensional Complex Analysis, Aizenberg and Yuzhakov, AMS, 1982]

Trivial case: with a strong reference wave added to F, i.e. 1 + F then ~ exp(F) for |F| << 1 and FT{1 + F} = δ + f; thus we expect to recover f when a (strong) reference point is present somewhere in the object domain.

Reference waves and holography



Importance of Phase and Reference Waves

Significance of phase and amplitude in the Fourier domain

Adolf W. Lohmann,* David Mendlovic, and Gal Shabtay

Department of Physical Electronics, Faculty of Engineering, Tel Aviv University, 69978 Tel Aviv, Israel

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We add new thoughts and aspects to the importance of phase and amplitude in the Fourier domain. We show how very similar objects react radically differently if, in the Fourier domain, either the phase was lost completely or the amplitude was modified to be constant. We also discuss the great influence of symmetry on the relative significance of the Fourier amplitude and of the Fourier phase. We show how changing the value of one pixel in some objects completely changes the significance of the Fourier phase and amplitude. [S0740-3232(97)00111-7]

JOSA A 14, p2901, 1997

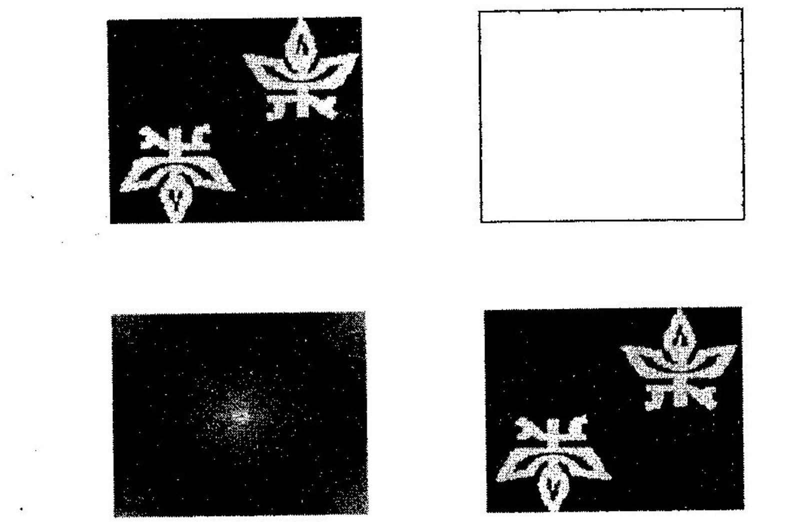


Fig. 8. Reconstructions from an asymmetric object with totally random phase: peak phase zero amplitude-only reconstruction (upper left), peak phase zero phase-only reconstruction (upper right), peak phase 90° amplitude-only reconstruction (lower left), peak phase 90° phase-only reconstruction (lower right).

See also Millane and Hsiao, *"On apparent counterexamples to phase dominance"*, JOSA A 20, p753, 2003

reference point ("peak") not shown

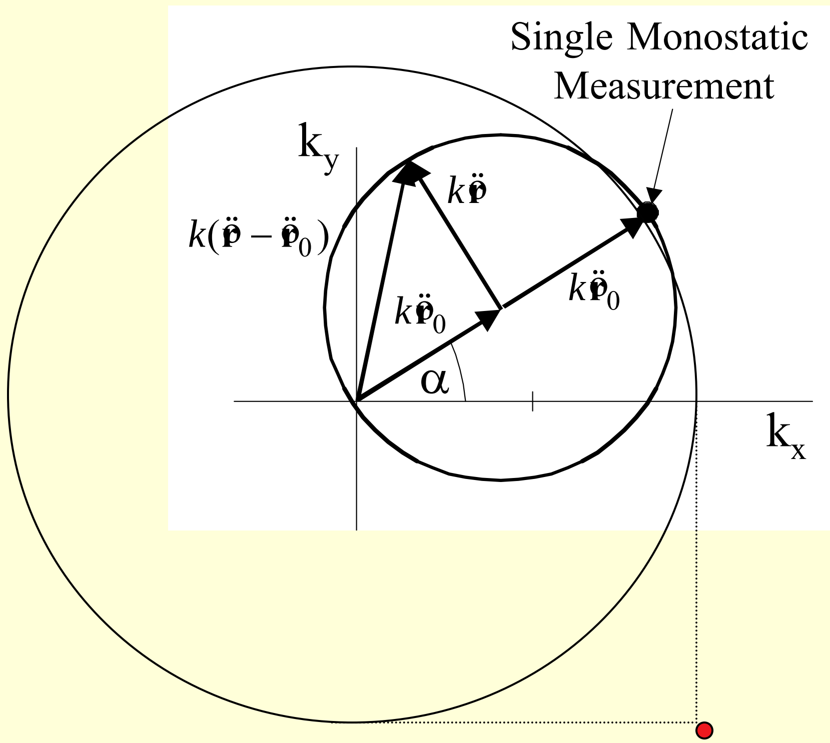
2D factorization and inverse scattering

Consider VΨ recovered (approximately) from k-space data

FT and spatially filter

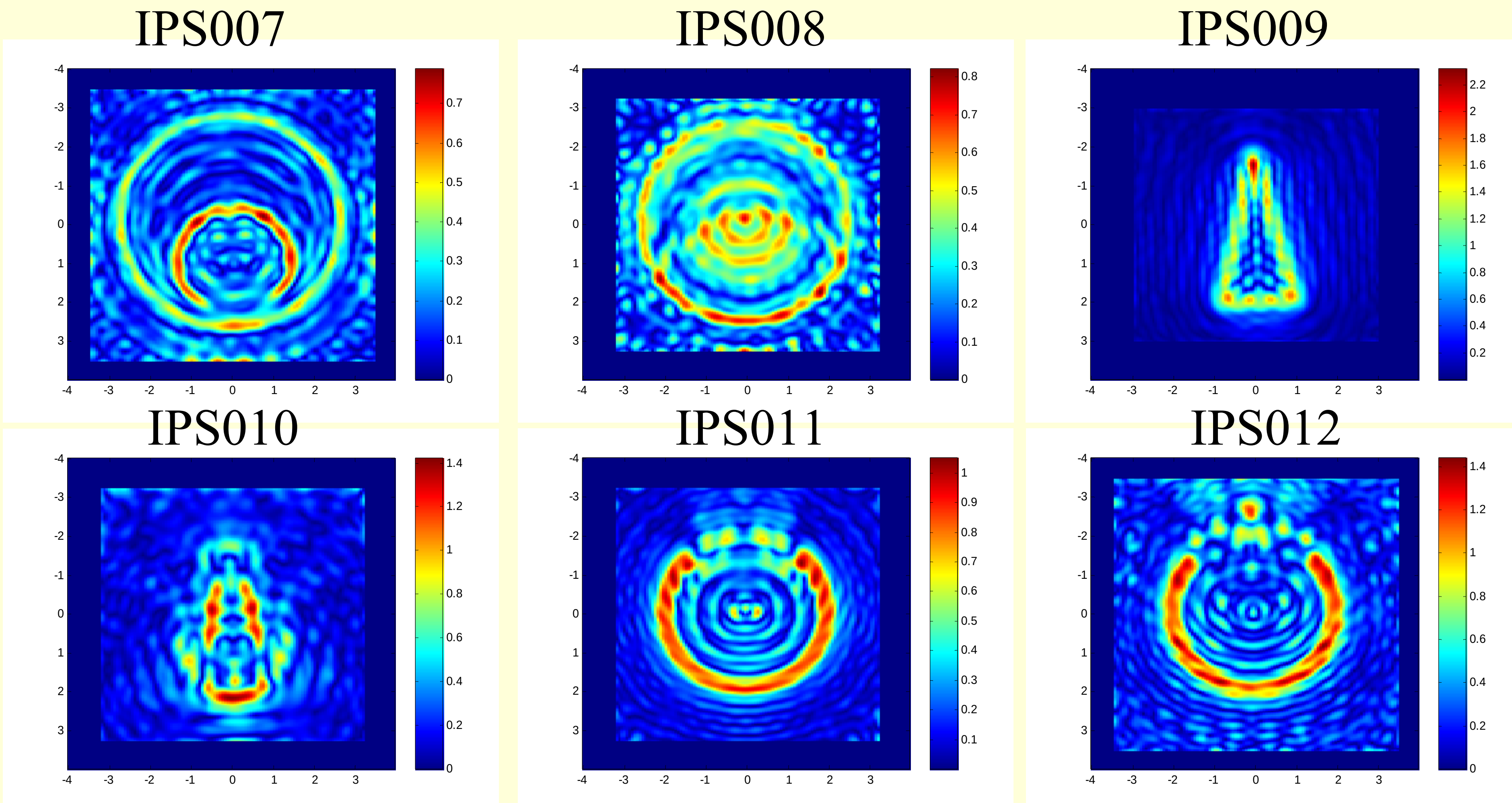
linear phase factor added to VΨ

reference point in k-space



Probing penetrable targets

Image Estimates of Ipswich Targets



Conclusions

Physical fields are strongly constrained by their analytic properties and these properties can be exploited.

The analytic properties of the field dictate zero properties and the zeros encode information.

The study of the zero trajectories of diffraction patterns has lead to the relatively new field of "singular optics" **(see e.g. Soskin and Vasnetsov, ch 4, Prog. in Optics, 42, 2001, Ed. E. Wolf)**

Solutions to phase retrieval, superresolution and inverse scattering problems are intimately tied to the properties of the function's zeros.

Minimum phase properties are important and the incorporation of a reference wave can accomplish this.

Appendix 1:Hilbert Transform

Since

exp(G) = exp(Re{G}).exp(iIm{G})

= exp(S.cos(cx)).exp(iS.sin(cx))

= |F| exp(iθ)

where Im{G}) = θ {= S sin(cx) in simple example)

We require HT to transform Scos(cx) to iSsin(cx) ….hence:

Let cepstrum = g

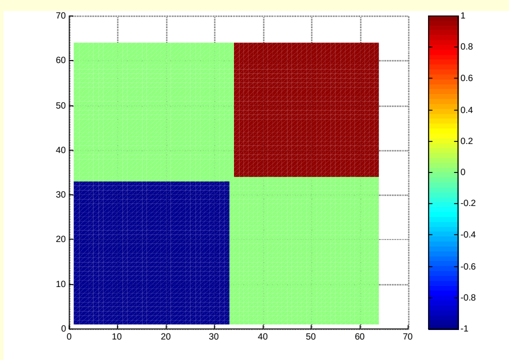
= s ⊗ δ(□causal □□□□□□)

then FT{g} = G = |G| exp(iθ)

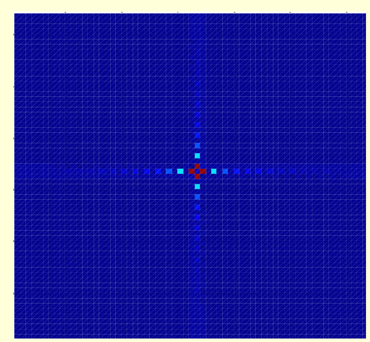
= Re{G} + iIm{G}

= S * exp(i[causal shift]x)

≡ S*exp(icx)



HT filter [not i.sgn(x).sgn(y)]



Point spread function

Appendix 2: What do we mean by phase?

$F(x,y) = |F(x,y)| \exp[\square \phi(x,y)]$

Recent clarification:

Emil Wolf "*Significance and measurability of the phase of a spatially coherent optical field*" Opt. Lett. 28 p5 2003

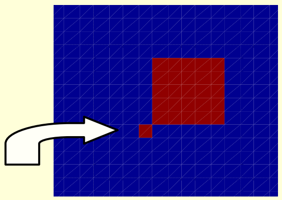
Phase usually considered in context of monochromatic wave

….but fields have finite bandwidth and phase fluctuates rapidly

Ability to interfere is measure of coherence and nonmonochromatic fields can be completely spatially coherent.

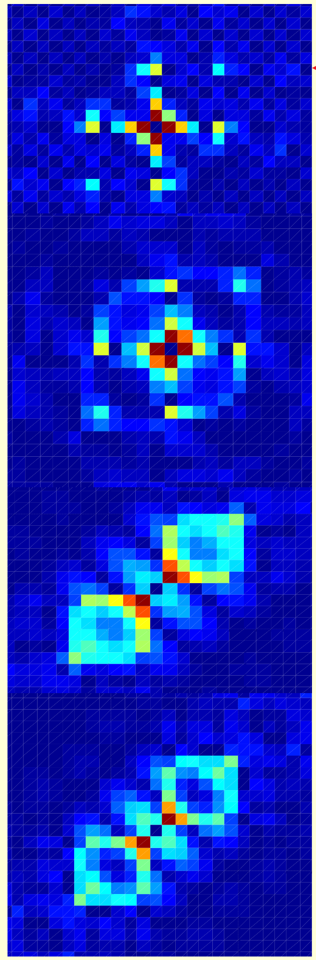
We can associate with any field that is spatially coherent at ω, a monochromatic field of the same frequency that yields the cross-spectral density of the field.

Encoding phase information



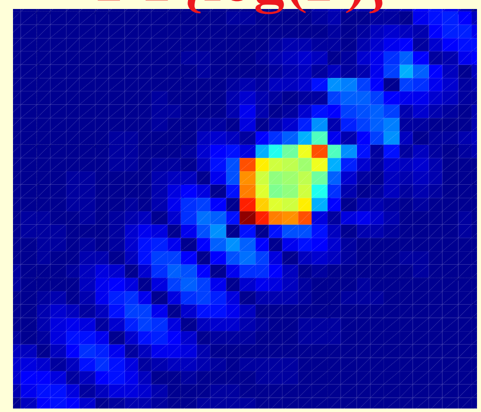
reference point:

amplitude



FT{log(Fourier intensity)}

FT{log(F)}



A test: if FT{log(I)} can be approximately separated then F is probably close to being minimum phase!

Trivial case: with a strong reference wave added to F, i.e. 1 + F then ~ exp(F) for |F| << 1 and FT{1 + F} = δ + f; thus we expect to recover f when a (strong) reference point is present somewhere in the object domain.